Topology Vol. 26, No. 4, pp. 475–504, 1987
Printed in Great Britain.

0040-9383/87 $3.00 + .00

TOPOLOGY OF COMPLEX WEBS OF CODIMENSION ONE AND GEOMETRY OF PROJECTIVE SPACE CURVES

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(*Received 7 December 1984*)

Dedicated to Professor H. Toda on his 60th birthday.

A b-WEB of a manifold M of codimension 1 is a configuration $\mathscr{W}$ of b foliations $\mathscr{F}_1, \ldots, \mathscr{F}_b$ of M of codimension 1. In §1, we prove that the topological and analytic classifications are the same for complex analytic webs of a complex manifold M under the condition $b \geqq \dim M + 1$ and a certain generic condition (Theorem 1.5.1). This is a complex analytic version of Dufour's theorem for C^∞-webs [15, 16]. In §2, we apply this theorem to the d-webs $\mathscr{W}_C$ of the dual projective space $\mathbb{P}_n^\vee$ of codimension 1 generated by the dual hyperplanes $x^\vee \in \mathbb{P}_n^\vee$ of $x \in C$ for algebraic curves $C \subset \mathbb{P}_n$ of degree d, and prove that the imbeddings $C \subset \mathbb{P}_n$ are determined by the topological structures of $\mathscr{W}_C$ up to projective transformations if $d \geqq n+2$ (Theorem 2.1.3). The singular locus $\Sigma(\mathscr{W}_C)$ of $\mathscr{W}_C$ is closely related to the projective geometry of C and the dual variety and curve of C. In the final two sections, we investigate the structure of $\mathscr{W}_C$ for the exceptional cases that $C \subset \mathbb{P}_n$ is of degree $n, n+1$, e.g. rational and elliptic normal curves, and singular plane curves.

A foliation $\mathscr{F}_i$ of a manifold M is locally defined to be a family of level surfaces of non-singular functions u_i on M, so the local study of b-webs is equivalent to one of the diagrams of functions of the form:

$$\begin{tikzcd} M \arrow[r, "u_i"] \arrow[dr] & \mathbb{K} \arrow[d, dash, dashed] \\ & \mathbb{K} \end{tikzcd} \qquad (\mathbb{K} = \mathbb{R}, \mathbb{C}).$$

The diagram of this type appears often in various areas of differential topology and its applications. In particular the envelope theory is reformulated by the diagram of this type and was studied by Thom [27], Arnol'd [1], Carneiro [9], Dufour [16] and Bruce [8].

The problem of this diagram is the simplest and a very attractive part of the general theory of diagram of C^∞-mappings, for which Thom–Mather theory does not work properly because of the fact that Malgrange's preparation theorem fails [13]. This difficulty seems not to be only on account of the appearance the diagram: in fact Dufour [13, 14] proved that for non-degenerate diagrams of three functions F, G: $\begin{matrix} \mathbb{R}^2 \rightarrow \mathbb{R} \\ \searrow \mathbb{R} \\ \searrow \mathbb{R} \end{matrix}$ (or $\begin{tikzcd} \mathbb{R}^2 \arrow[r] \arrow[dr] & \mathbb{R}^2 \\ & \mathbb{R} \end{tikzcd}$), F, G are C^∞-equivalent if and only if topologically equivalent (Lemma 1.1.1) basically by using Lebesque's theorem, and consequently that the topological stability theorem does not hold for these divergent diagrams. This is in contrast to the known result that for the convergent diagrams of C^∞-mappings: $\begin{matrix} \searrow \\ \nearrow \searrow \\ \searrow \nearrow \\ \nearrow \end{matrix} \rightarrow$, the Thom–Mather theory works properly and the topological stability theorem holds [4, 12, 14, 24].

In §1 we prove a Dufour-type theorem for the complex analytic case, namely *if two 3-webs $\mathscr{W}=(\mathscr{F}_1,\mathscr{F}_2,\mathscr{F}_3)$, $\mathscr{W}'=(\mathscr{F}'_1,\mathscr{F}'_2,\mathscr{F}'_3)$ of a complex 2 manifold M are non-degenerate and non-hexagonal (i.e. $\mathscr{W}$, $\mathscr{W}'$ are nowhere equivalent to the germ of 3-web of foliations by parallel lines in $\mathbb{C}^2$), and there is a homeomorphism h of M such that $h(\mathscr{W}_i)=\mathscr{W}'_i$, then h is holomorphic or anti-holomorphic.* This theorem is proved in a more general form (Theorem 1.5.1).

Our proof goes through by purely topological analysis of the holonomy map (see §1.2) which is a map from a leaf onto itself defined by the rotation by walking along the strings of the web around a fixed center, passing across the three warps through the center (see Fig. 1). With the level function u_i for some i, this mapping is of the form: $P_i^2: x \to x - kx^3 + \ldots$, *where* k is the web curvature of $\mathscr{W}$ at the center of rotation (Proposition 1.3.1). The stable and unstable sets $S(P_i^2, 2x)$, $U(P_i^2, 2x)$ are defined similarly to the ordinary stable and unstable manifolds of endomorphisms (see §1.4). They are real lines $\mathbb{R}/\sqrt{k}$, $\sqrt{-1}\ \mathbb{R}/\sqrt{K}$ forming a right angle with each other in leaves at the center of rotation if $k \neq 0$ (Proposition 1.4.1). Since these real lines are constructed pure geometrically, a detailed argument shows that the homeomorphism $h:\mathscr{W}\to\mathscr{W}'$ is differentiable and ± 1 conformal. Thus h is holomorphic or anti-holomorphic respectively if orientation preserving or not. The condition $k \neq 0$ is essential, for our analysis.

Classically it is known that a 3-web of a real or complex surface is hexagonal if and only if the web curvature is identically zero [7], so the condition that $\mathscr{W}$, $\mathscr{W}'$ are non-hexagonal is necessary. In fact any hexagonal 3-webs of foliations by parallel lines in $\mathbb{C}^2$ admit real linear but not complex analytic automorphisms.

In §2, we apply the theorem to the dual d-webs $\mathscr{W}_C^\vee$ generated by algebraic curves $C \subset \mathbb{P}_n$ of degree d. The d-web $\mathscr{W}_C^\vee$ has the singular locus $\Sigma(\mathscr{W}_C^\vee)=\mathrm{envl}(\mathscr{W}_C^\vee) \cup \mathrm{degn}\,(\mathscr{W}_C^\vee)$ [see §2 (A)] outside which $\mathscr{W}_C^\vee$ forms a non-degenerate d-web. In the case of $n=2$, Graf-Sauer's theorem says that $\mathscr{W}_C^\vee$ is hexagonal outside $\Sigma(\mathscr{W}_C^\vee)$ if and only if $C \subset \mathbb{P}_2$ is a cubic curve (Theorem 2.3.1, or see [7]). This result was expanded by many authors [2, 3, 5, 11].

The restriction of $\mathscr{W}_C^\vee$ to an intersection $x_1^\vee \cap \ldots \cap x_{n-2}^\vee = \mathbb{P}_2$, $x_i \in C$ is the web generated by the image of C under the projection of $\mathbb{P}_n$ with the center $\mathbb{P}_{n-3}$ spanned by $x_1, \ldots, x_{n-2}$. If $C \cdot \mathbb{P}_{n-3} = x_1 + \ldots + x_{n-2}$ is non-singular, the image is a plane curve of degree $d-n+2$ (Proposition 2.1.4). Therefore the theorem applies to restrictions of $\mathscr{W}_C^\vee$ to generic planes $\mathbb{P}_2 = x_1^\vee \cap \ldots \cap x_{n-2}^\vee \subset \mathbb{P}_n$ if the degree $d \geqq n+2$ and we get:

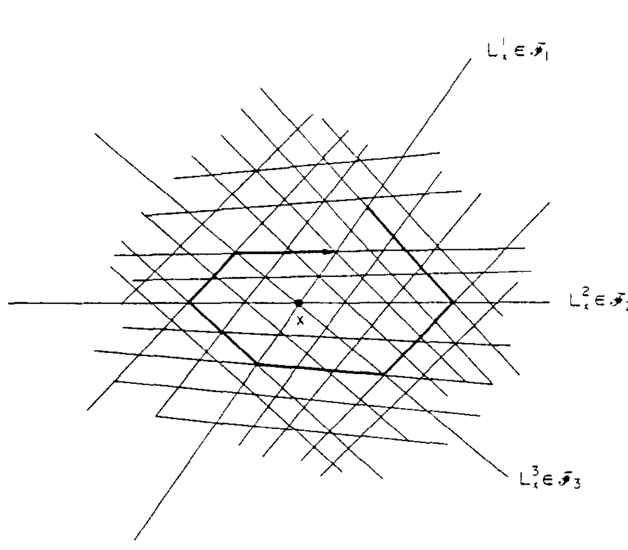

Fig. 1. The holonomy map of a 3-web $\mathscr{W}=(\mathscr{F}_1,\mathscr{F}_2,\mathscr{F}_3)$ of $\mathbb{C}^2$ with the center x.

THEOREM 2.1.3. *Let $C, C' \subset \mathbb{P}_n$ be irreducible algebraic curves of degree $\geqq n+2$ and let h be a homeomorphism of the dual space $\mathbb{P}_n^\vee$ such that $h(\mathscr{W}_C) = \mathscr{W}_{C'}$. Then h or its complex conjugate $\bar{h}$ is a projective linear transformation of $\mathbb{P}_n^\vee$.*

Corollary 2.1.5 says roughly that *the complex structure of a line bundle $L \to C$ on a Riemann surface is determined by the topological structure of a net of effective divisors determining L.*

In §2.2, we investigate some results on the geometry of the singular locus $\Sigma(\mathscr{W}_C)$, some of which are classically known and can be found in [19, 26, 29]. A point y lies in $\Sigma(\mathscr{W}_C)$ if and only if $y^\vee \cdot C$ is singular or an n-tuple of points in $y^\vee \cdot C$ does not span $y^\vee = \mathbb{P}_{n-1}$. Corresponding to the multiplicity and degeneracy of $y^\vee \cap C$, we define the filtration $P_n^\vee = \mathrm{envl}^0\ (\mathscr{W}_C) \supset \mathrm{envl}^1\ (\mathscr{W}_C) \supset \dots \supset \mathrm{envl}^{n-1}\ (\mathscr{W}_C) \supset \dots$ and $\mathrm{degn}\ (\mathscr{W}_C)$ so that $\mathrm{envl}^1\ (\mathscr{W}_C) \cup \mathrm{degn}\ (\mathscr{W}_C) \cup (\mathrm{sing}\ C)^\vee = \Sigma(\mathscr{W}_C)$. The sets $\mathrm{envl}^1\ (\mathscr{W}_C)$, $\mathrm{envl}^{n-1}\ (\mathscr{W}_C)\ (= C^\vee)$ are called the *dual variety* and *dual curve* of C respectively, and $\mathrm{envl}^{i-1}\ (\mathscr{W}_C) = \tan(\mathrm{envl}^i(\mathscr{W}_C))$ (Proposition 2.2.2). $\mathrm{Envl}^i\ (\mathscr{W}_C)$ *is the union of the osculating $n-i-1$ planes of $C^\vee$*. From the duality of the osculating i bundle of C and $n-i-1$ bundle of $C^\vee$, it follows that $\mathrm{envl}^i\ (\mathscr{W}_C)$ *and* $\mathrm{envl}^{n-i-1}\ (\mathscr{W}_C)$ *are dual to each other* (Proposition 2.2.1).

The structure of the set $\mathrm{degn}\ (\mathscr{W}_C)$ is determined by the various secant varieties of C, but their structure seems to be unknown even for simple space curves.

Section 2.3 is devoted to an introduction of relations of the quasi-group structure of C and the geometry of the web $\mathscr{W}_C$, and Graf-Sauer's theorem.

In §2.4 and §2.5 we report the web structure for the exceptional cases $d = n, n+1$ of Theorem 2.1.3. First in §2.4, we consider the cases that $C \subset \mathbb{P}_n$ is a non-singular curve of degree $n, n+1$, i.e. the rational or elliptic normal curve of degree n or $n+1$, respectively.

The geometry of elliptic normal curves C_{n+1} of degree $n+1$ has been historically studied. We recall from [20] the $\mathbb{Z}_{n+1} \times \mathbb{Z}_{n+1}$ symmetry of C_{n+1} which is induced from the representation of the Heisenberg group H_{n+1} on $\mathbb{C}^{n+1}$. Theorem 2.1.3 suggests that $\mathscr{W}_{C_{n+1}}$ may have a stronger topological symmetry than $\mathbb{Z}_{n+1} \times \mathbb{Z}_{n+1}$. Using the group structure of the elliptic curve C_{n+1} and Abel's theorem, we prove that the semi-direct product $GL(2, \mathbb{Z}) \ltimes (\mathbb{Z}_{n+1} \times \mathbb{Z}_{n+1})$ acts on $\mathscr{W}_{C_{n+1}}$ as homeomorphisms of $\mathbb{P}_n$ (Proposition 2.4.4). The fact that $n+1$ torsion points of C_{n+1} are hyper-osculating points is already known by e.g. [21] and the degree of $\mathrm{envl}^i\ (\mathscr{W}_{C_{n+1}})$ is presented, as a consequence of a general formula by Piene [26].

Any curve of genus 1, 0 and degree $n+1, n$ in $\mathbb{P}_m$ is given by projecting the elliptic, rational normal curve of degree $n+1, n$ from a generic center. This corresponds, in turn to their webs, to the restrictions of $\mathscr{W}_C$ to the $n-m-1$ plane dual to the center (cf. Proposition 2.1.4). This might be of some use for the study of those curves.

By the duality of curves C and $C^\vee = \mathrm{envl}^{n-1}\ (\mathscr{W}_C)$, $\mathscr{W}_C$ is reproduced from $C^\vee$, so we can say $\mathrm{envl}^i(\mathscr{W}_C)$ all have faithful information of the original web $\mathscr{W}_C$. So we are led to the geometry of $\mathrm{envl}^1(\mathscr{W}_C)$. From another point of view, we can regard $\mathbb{P}_n^\vee$ as the parameter space of the deformation $C \cdot y^\vee$, $y \in \mathbb{P}_n^\vee$, and then $\mathrm{envl}^1\ (\mathscr{W}_C)$ is the discriminant (bifurcation) set of the deformation.

In §2.5 we list results for singular plane cubic curves.

Last of all the author would note that the motivation for this paper was originally a topological classification of non-singular vector bundle mappings of bundles of rank $n-1$ to those of rank n. In another paper [25], the author proved that the topological structure of generic involutive mappings $f: \bar{N} \to \bar{P}$ of involutive manifolds is determined by the differential

$$\begin{array}{ccc} df\colon \mathscr{N}_N & \to & \mathscr{N}_P \\ \downarrow & & \downarrow \\ f\colon N & \to & P \end{array}$$

of the normal bundles of the fixed point sets $N \subset \bar{N}$, $P \subset \bar{P}$, under a certain condition. The results in §2 offer a partial answer to this problem.

This paper was written under the supervision of Professor M. Adachi and Professor H. Toda while the author was in Kyoto University.

§1. A TOPOLOGICAL RIGIDITY THEOREM FOR COMPLEX WEBS OF CODIMENSION ONE

1. *Preliminaries in web geometry*

Let M be a C^r manifold of dimension m. $r=0,\ldots,\infty$ or ω, i.e. real or complex analytic. We call a b-tuple $\mathscr{W}=(\mathscr{F}_1,\ldots,\mathscr{F}_b)$ of C^r foliations of M of codimension 1 a *b-web of M of codimension* 1, and we say $\mathscr{W}$ is *non-degenerate* if $\mathscr{F}_i$ are in general position. We call a subtuple $(\mathscr{F}_{i_1},\cdots,\mathscr{F}_{i_c})$ *a subweb of* $\mathscr{W}$. Two b-webs $\mathscr{W}=(\mathscr{F}_1,\ldots,\mathscr{F}_b)$, $\mathscr{W}'=(\mathscr{F}'_1,\ldots,\mathscr{F}'_b)$ are *C^s equivalent* if there is a C^s diffeomorphism h of M such that $h(\mathscr{F}_i)=\mathscr{F}'_i$ for $i=1,\ldots,b$. Then we denote $h(\mathscr{W})=\mathscr{W}'$.

LEMMA 1.1.1. [13, 14]. *Let $\mathscr{W}$, $\mathscr{W}'$ be non-degenerate C^r-m+1-webs of a real C^r-m-manifold M of codimension 1 and let h be a homeomorphism of M such that $h(\mathscr{W})=\mathscr{W}'$. Then h is a C^r diffeomorphism of M for $r=\infty,\omega$. This holds also for germs of $m+1$-webs.*

A C^r–b-web $\mathscr{W}$ is *octahedral* (*hexagonal* for $m=2$) if $\mathscr{W}$ is everywhere locally C^r equivalent to a b-web of $\mathbb{R}^m$ (or $\mathbb{C}^m$) by foliations with parallel hyperplanes as leaves. In other words, $\mathscr{W}$ is octahedral if $\mathscr{W}$ is everywhere locally C^0 equivalent to the octahedral b-web consisting of hyperplanes, by Dufour's theorem (Lemma 1.1.1) in the real case of $m+1\leq b$, $r=\infty$. This equivalence of definitions was already known in [7]. In the following we introduce the classical results of web geometry following [7] for the case $n=2$.

Let $\mathscr{W}=(\mathscr{F}_1,\mathscr{F}_2,\mathscr{F}_3)$ be a non-degenerate C^r–3-web of a C^r surface M defined by non-singular C^r–1-forms $\omega_1,\omega_2,\omega_3$ with $\omega_1+\omega_2+\omega_3=0$ and $r=3,\ldots,\infty,\omega$. Then

$$\Omega=\omega_1\wedge\omega_2=\omega_2\wedge\omega_3=\omega_3\wedge\omega_1$$

holds and Ω is non-singular. Define the functions h_i by

$$\mathrm{d}\omega_i=h_i\Omega,$$

for $i=1,2,3$. Then we have

$$\gamma=h_3\omega_2-h_2\omega_3=h_1\omega_3-h_3\omega_1=h_2\omega_1-h_1\omega_2$$

and

$$\mathrm{d}\omega_i=\gamma\wedge\omega_i,$$

for $i=1,2,3$. Define the function k on M by

$$\mathrm{d}\gamma=k\Omega.$$

Using local coordinates x_1, x_2 of M, we have

$$k=h_{2,1}-h_{1,2}=h_{3,2}-h_{2,3}=h_{1,3}-h_{3,1},$$

where $h_{i,j}=\partial/\partial x_j h_i$.

It is easy to see that the 2-form $\mathrm{d}\gamma=k\Omega$ is independent of the choice of 1-forms $\omega_1,\omega_2,\omega_3$ defining the foliations $\mathscr{F},\mathscr{F}_2,\mathscr{F}_3$, but dependent only on the web $\mathscr{W}$ [6, §6–8]. We call k,Ω, $\mathrm{d}\gamma=k\Omega$ as follows:

k *web curvature of* $\mathscr{W}$

Ω *surface element of* $\mathscr{W}$

$\mathrm{d}\gamma=k\Omega$ *normalized surface element of* $\mathscr{W}$.

Let x,y be local coordinates of M, u_i the local level C^r functions defining $\mathscr{F}_i$ and let W be a C^r–function such that

$$W(u_1,u_2,u_3)=0.$$

We call W a *web function of* $\mathscr{W}$ (or u_1, u_2, u_3).

Let $W_{i,j,k} = \partial^3/\partial u_i \partial u_j \partial u_k W$ and $\omega_i = W_i \cdot du_i$ for $i, j, k = 1, 2, 3$. Then $k, d\gamma, \Omega$ are calculated as follows:

$$\Omega = W_1 W_2 \cdot du_1 \wedge du_2 = W_2 W_3 \cdot du_2 \wedge du_3 = W_3 W_1 \cdot du_3 \wedge du_1$$

$$d\gamma = \frac{1}{2} \sum_{r,s=1}^{3} \frac{\partial^2}{\partial u_r \partial u_s} \cdot \log \frac{W_r}{W_s} \cdot du_r \wedge du_s,$$

$$k = A_{2,3} + A_{3,1} + A_{1,2},$$

$$A_{i,j} = \frac{1}{W_r W_s} \cdot \frac{\partial^2}{\partial u_r \partial u_s} \log \frac{W_r}{W_s}$$

$$= \frac{W_{rrs}}{W_r^2 W_s} - \frac{W_{rss}}{W_r W_s^2} + \frac{W_{rs}}{W_r W_s}\left(\frac{W_{ss}}{W_s^2} - \frac{W_{rr}}{W_r^2}\right).$$

THEOREM 1.1.2. *Let* $\mathscr{W}$ *be a non-degenerate* C^r*-3-web of a* C^r *surface* M *and* $r = 3, 4, \ldots, \infty, \omega$ *(real or complex analytic). Then* $\mathscr{W}$ *is hexagonal if and only if the normalized surface element* $k\Omega$ *(or the web curvature* k*) is identically zero on* M.

For the proof of this theorem, see e.g. [6]. This result was expanded by many authors (see [7, 10]).

The geometric meaning of the web curvature k is explicitly explained in Section 1.3.

2. *Maps associated with webs: holonomy map*

The geometric structure of a web is translated into the structure of the translation maps $T^{j,k}_{p,q}$ between two leaves along leaves passing through them transversally (see Fig. 2). These translation maps yield many topological invariants.

In this section we study non-degenerate analytic 3-webs of an open neighbourhood U of $0 \in \mathbb{C}^2$, $\mathscr{W} = (\mathscr{F}_1, \mathscr{F}_2, \mathscr{F}_3)$ defined by level functions u_i: $(\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$. We define $u_{i-3n} = u_i$, $n \in \mathbb{Z}$, $i = 1, 2, 3$ as a convention. Let $L^i_p = \{p' \in U \mid u_i(p') = u_i(p)\}$ denote the leaf of $\mathscr{F}_i$ passing through the point $p \in U$. For a point $q \in L^i_p$ and $j, k \neq i$, the *translation map* $T^{j,k}_{p,q}$: $(L^j_p, p) \to (L^k_q, q)$ is defined by

$$T^{j,k}_{p,q} = (u_i \mid L^k_q)^{-1} \, (u_i \mid L^j_p).$$

This construction of $T^{j,k}_{p,q}$ is recovered by the geometry of $\mathscr{W}$: $T^{j,k}_{p,q}(r) = s$ if $L^i_r \cap L^k_q = \{s\}$, and assuming $L^i_p \subset U$ is connected, the germ $T^{j,k}_{p,q}$ at p is independent of the choice of level functions u_i. We denote $T^{j,k}_{p,p}$ as $T^{j,k}_p$. Clearly we have

$$T^{j,k}_{p,q} \circ T^{k,j}_{q,p} = \mathrm{id}$$

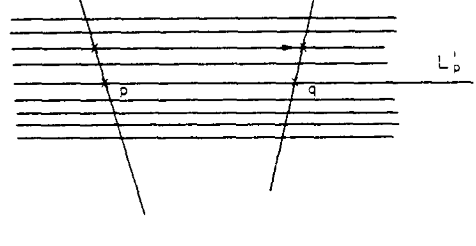

Fig. 2. The translation map $T^{j,k}_{p,q}$.

and

$$u_i \circ T^{j,k}_{p,q} = \mathrm{id}.$$

Next we define the germ (holonomy map or Poincaré map) by

$$P^{i,j,k}_p = T^{k,i}_p \circ T^{j,k}_p \circ T^{i,j}_p \colon (L^i_p, p) \to (L^i_p, p),$$

for a distinct triple (i, j, k). We denote $P^{i,i+1,i+2}_p$ simply by P^i_p. By definition we see that

$$P^{i,j,k}_p \circ P^{i,k,j}_p = \mathrm{id}$$

and

$$P^{i+1}_p \circ T^{i,i+1}_p = T^{i+3,i+4}_p \circ P^i_p,$$

from which we have

$$(u_{i+2}|L^{i+1}_p) \circ P^{i+1}_p \circ (u_{i+2}|L^{i+1}_p)^{-1} = (u_{i+2}|L^i_p) \circ P^i_p \circ (u_{i+2}|L^i_p)^{-1},$$

which we denote simply by

$$\bar{P}^{i+2}_p \colon (\mathbb{C}, u_{i+2}(p)) \to (\mathbb{C}, u_{i+2}(p)).$$

We define $\bar{T}^{i,j}_p \colon (\mathbb{C}, u_i(p)) \to (\mathbb{C}, u_j(p))$ by

$$\bar{T}^{i,j}_p = (u_j|L^k_p) \circ T^{j,k}_p \circ (u_i|L^j_p)^{-1}.$$

Then we have

$$\bar{P}^i_p = \bar{T}^{i+2,i+3}_p \circ \bar{T}^{i+1,i+2}_p \circ \bar{T}^{i,i+1}_p.$$

Next we introduce a local translation map of $\mathbb{C}$: the range of u_i. For a small $s \in \mathbb{C}$ we define the points $q, r \in \mathbb{C}^2$ by $u_j(p) = u_j(q)$, $u_k(q) = u_k(r)$, $u_i(r) = u_i(p)$ and $u_j(r) = u_j(p) + s$. We define the local translation map $A^{j,k}_{p+s} \colon (\mathbb{C}, u_j(p)) \to (\mathbb{C}, u_j(p) + s)$ by

$$A^{j,k}_{p+s} = u_j \circ T^{i,i}_{q,r} \circ T^{i,i}_{p,q} \circ (U_j|L^i_p)^{-1}$$

for $j \neq k$ (see Fig. 3). We denote this sometimes as

$$A^{j,k}_{p+s}(t) = t +^{j,k}_p s.$$

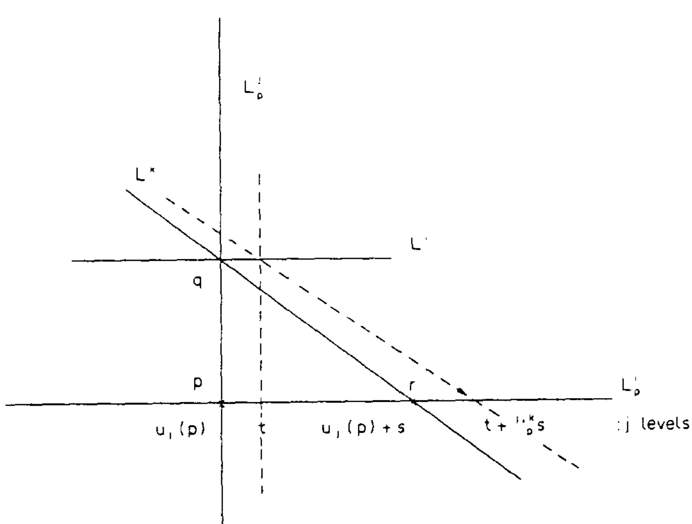

Fig. 3. The local translation map $A^{j,k}_{p+s} = +^{j,k}_p s$.

By an infinitesimal calculation we see that

$$A^{j,k}_{p+s}(u_j(p)+t)=u_j(p)+t+s+R_2(t),$$

where R_2 denotes the remainder terms of order $\geqq 2$. Note that

$$A^{j,k}_{p+s}\circ A^{j,k}_{p-s}=\mathrm{id}.$$

For a point $p\in U$, we define the mapping C^i_p: $(\mathbb{C},u_i(p))\to(\mathbb{C},u_i(p))$ by

$$C^i_p(u_i(p)+t)=u_i\circ(u_j,u_k)^{-1}(u_j(u_i,u_k)^{-1}(u_i(p)+t,u_k(p)),u_k(u_j,u_i)^{-1}(u_j(p),u_i(p)+t))$$

and $C^{i,j}_p$: $(L^j_p,p)\to(L^j_p,p)$ by

$$C^{i,j}_p=(u_i|L^j_p)^{-1}\circ C^i_p\circ(u_i|L^j_p).$$

It is easy to see that $C^{i,j}_p$ is independent of the choice of the level functions u_i.

By an infinitesimal calculation, we see that C^i_p is of the form:

$$C^i_p(u_i(p)+t)=u_i(p)+2t+R'_2(t),$$

where R'_2 denotes the remainder terms of order $\geqq 2$.

3. *Calculation of the Poincare map*

We study a non-degenerate analytic 3-web of an open neighbourhood U of $0\in\mathbb{C}^2$. First we assume that the level functions and the web function are of the following form:

$$(*)\qquad \begin{aligned} u_1&=x,\\ u_2&=y,\\ u_3(x,y)&=\omega(x,y)=x+y+a(x^2y-xy^2)+R_4(x,y) \end{aligned}$$

and

$$W(t_1,t_2,t_3)=\omega(t_1,t_2)-t_3,$$

where R_4 denotes the remainder terms of order $\geqq 4$ such that $R_4(t,0)=R_4(0,t)=0$ and $R_4(t,t)=0$. Then

$$L^1_0=y\text{-axis},\ L^2_0=x-\text{axis and } L^3_0=\{\omega(x,y)=0\}.$$

Let $(0,y)\in L^1_0$. By the normal form $(*)$, we see that

$$T^{1,2}_0(0,y)=(y,0).$$

Let $T^{2,3}_0(y,0)=(y,y_1)$. Then, by the equality $W(u_1,u_2,u_3)=\omega(u_1,u_2)-u_3$ at the point (y,y_1), we have

$$\begin{aligned} W(u_1,u_2,u_3)&=\omega(u_1,u_2)-0\\ &=y+y_1+a(y^2y_1-yy_1^2)+R_4(y,y_1)=0. \end{aligned}$$

Thus we have

$$\begin{aligned} y_1&=-y-a(y^2(-y)-y(-y)^2)+R'_4(y)\\ &=-y+2ay^3+R'_4(y). \end{aligned}$$

Clearly

$$T^{3,1}_0(y,y_1)=(0,y_1),$$

so we have

$$\text{(a)}\qquad P^{1,2,3}_0(0,y)=(0,y_1),$$

and similarly

$$P^{2,3,1}(x,0)=(x_1,0),$$
$$x_1=-x+2ax^3+R'_4(x),$$

hence we have

$$\bar{P}_0^3(t)=-t+2at^3+R''_4(t).$$

Next we consider general u_i and W. Let $W_i(0,0,0)=a_i$, $i=1,2,3$, and $\{W=0\}=\{\omega(t_1,t_2)+a_3t_3=0\}$ with an analytic function ω on $\mathbb{C}^2$ such that $\omega_i(0,0)=a_i$, $i=1,2$. Then

$$(**)\qquad W(t_1,t_2,t_3)=f(t_1,t_2,t_3)\cdot(\omega(t_1,t_2)+a_3t_3)$$

with an analytic function f with $f(0,0,0)=1$. Let $u'_1(t)=\omega(t,0)$, $u'_2(t)=\omega(0,t)$ and define the functions f' and ω' by the next commutative diagram:

$$\begin{array}{rcc} f,\omega\colon & \mathbb{C}^3 \longrightarrow & \mathbb{C} \\ {\scriptstyle (u'_1,u'_2,\mathrm{id})} & \downarrow & \| \\ f',\omega'\colon & \mathbb{C}^3 \longrightarrow & \mathbb{C}. \end{array}$$

Then $f'(0,0,0)=1$ and $\omega'(t,0)=\omega'(0,t)=t$.

By Poincaré's linearization lemma, the function $A(t)=\omega'(t,t)$ is conjugate with the linear function $L(t)=2t$ by a germ of diffeomorphism $h\colon(\mathbb{C},0)\to(\mathbb{C},0)$ such that $h'(0)=1$:

$$\omega'(h(t),h(t))=h(2t).$$

Define the functions f'' and ω'' by the next commutative diagram:

$$\begin{array}{rcc} f',\omega'\colon & \mathbb{C}^3 \longrightarrow & \mathbb{C} \\ {\scriptstyle (h,h,h)} & \downarrow & \| \\ f'',\omega''\colon & \mathbb{C}^3 \longrightarrow & \mathbb{C}. \end{array}$$

Then $f''(0,0,0)=1$, $\omega''(t,0)=\omega''(0,t)=t$ and $\omega''(t,t)=2t$. We can replace the level functions u_i with

$$u''_i=h\circ u'_i\circ u_i,\ i=1,2,$$
$$u''_3=-h(a_3u_3),$$

and the web function $W=f\cdot(\omega+a_3t_3)$ with

$$W''(t_1,t_2,t_3)=\omega''(t_1,t_2)-t_3.$$

Then we have

$$\text{(b)}\qquad W''(u''_1,u''_2,u''_3)=\omega''(u''_1,u''_2)-u''_3=0,$$

$$\partial u''_i/\partial u_i(0)=a_i,\ i=1,2,$$
$$\partial u''_3/\partial u_3(0)=-a_3,$$

and

$$\omega''(t_1,t_2)=t_1+t_2+a(t_1^2t_2-t_1t_2^2)+R_4(t_1,t_2),$$

with a number $a\in\mathbb{C}$. By (a), we have

$$(u''_l\,|\,L_0^i)\circ P_0^{i,i+1,i-2}\circ(u''_l\,|\,L_0^i)^{-1}(t)=-t+2at^3+R'_{4,l}(t),$$

$i=1,,2,3$, $l\neq i$. By this together with (b), we have

$$(u_l\,|\,L_0^i)\circ P_0^{i,i+1,i+2}\circ(u_l\,|\,L_0^i)^{-1}(t)=-t+2a\,a_l^2t^3+R''_{4,l}(t).$$

Therefore we proved that for any point $(x, y) \in U$,

(c) $$(u_l | L^i_{(x,y)}) \circ P^{i,i+1,i+2}_{(x,y)} \circ (u_l | L^i_{(x,y)})^{-1}(u_l(x,y)+t) = u_l(x,y) - t + k_l(x,y)\, t^3 + R''_4(t),$$

where k_l is a function on $U, l \neq i$. In the following we will calculate the function k_l.

By a direct calculation with the form (**), we see that the web curvature of $\mathscr{W}$ (see introduction) is

$$\begin{aligned} k(W)\,(0,0) &= k(\omega + a_3 t_3)\,(0,0) \\ &= k(\omega' + a_3 t_3)\,(0,0) \\ &= 4a. \end{aligned}$$

Therefore we have, by (c),

$$k_l(0,0) = 2a\, a_l^2 = \tfrac{1}{2} k(W)(0,0) \cdot W_l^2(0,0,0).$$

Summarizing these results above, we have

PROPOSITION 1.3.1. *Let u_1, u_2, u_3 be level functions of a 3-web $\mathscr{W}$ of $\mathbb{C}^2$ and W a web function. Then*

$$\begin{aligned} &(u_l | L^i_{(x,y)}) \circ P^{i,i+1,i+2}_{x,y} \circ (u_l | L^i_{(x,y)})^{-1}(u_l(x,y)+t) \\ &\qquad = u_l(x,y) - t + \tfrac{1}{2} k(W)\,(x,y) \circ W_l^2(u_1,u_2,u_3)\,.\,t^3 + R_{4,l}(t), \end{aligned}$$

for $l \neq i$, where $R_{4,l}$ denotes the remainder terms of order $\geqq 4$.

Next we prove

PROPOSITION 1.3.2. *If a non-degenerate 3-web of an open neighborhood U of $0 \in \mathbb{C}^2$ with level functions u_i and a web function W is not hexagonal, i.e. the web curvature $k(W)$ is not identically zero on U, then $k(W) \cdot W_i^2(u_1,u_2,u_3)$ is not constant on a leaf L^i_p for some $i = 1,2,3$.*

Proof. For simplicity we suppose u_i and W are of the normal form (*) and $k(W)\,(0,0) = a = 1$ and $k(W) \cdot W_i^2(u_1,u_2,u_3)$ is constant on leaves L^i_p, $i = 1,2,3$. Then we have, on the leaf $\{u_1 = x_0\}$,

$$\begin{aligned} k(W)\,(x_0,y) \cdot W_1^2(x_0,y,\omega(x_0,y)) &= k(W)\,(x_0,0) \cdot W_1^2(x_0,0,x_0) \\ &= k(W)\,(x_0,0), \end{aligned}$$

and on the leaf $\{u_2 = y_0\}$,

$$\begin{aligned} k(W)\,(x,y_0) \cdot W_2^2(x,y_0,\omega(x,y_0)) &= k(W)\,(0,y_0) \cdot W_2^2(0,y_0,y_0) \\ &= k(W)\,(0,y_0), \end{aligned}$$

from which we have

$$k(W) = \frac{1}{W_1 W_2} \cdot \frac{\partial^2}{\partial x \partial y} \log\left(\frac{W_2}{W_1}\right) \equiv 0.$$

This contradicts the supposition $k(W)\,(0,0) = a = 1$.

Therefore we have proven the proposition.

4. *Characteristic sets of two function germs on $(\mathbb{C}, 0)$: stable and unstable sets*

Let $P, C\colon (\mathbb{C},0) \to (\mathbb{C},0)$ be germs of analytic functions with Taylor expansions $(P(z) = z - kz^3 + \ldots$ and $C(z) = 2z$. We define the germ $S(P,C)$, $U(P,C)$ at $0 \in \mathbb{C}$ to be the direct limits of the sets as follows:

$$S(P,C)=\varinjlim_{0\in U:\,\text{open}}\left\{\lim_{i\to\infty}\bar{C}^{j(i)}\bar{P}^{i}(z)\,|\,\bar{C}^{j}\bar{P}^{i}(z),\, z\in U,\, j\to\infty \text{ as } i\to\infty\right\},$$

$$U(P,C)=\varinjlim_{0\in U:\,\text{open}}\left\{\lim_{i\to\infty}\bar{C}^{j(i)}\bar{P}^{-i}(z)\,|\,\bar{C}^{j}\bar{P}^{-i}(z),\, z\in U,\, j\to\infty \text{ as } i\to\infty\right\},$$

where $\bar{C}, \bar{P}$ are representatives of C, P and $j(i)$ runs over the set of all sequences of positive integers such that the limit exists. Clearly these germs are dependent only on the germs C, P. By the definition we have $C(S(P,C))=S(P,C)$, $C(U(P,C))=U(P,C)$ and $S(P^{-1},C)=U(P,C)$, $U(P^{-1},C)=S(P,C)$.

The purpose of this section is to prove

Proposition 1.4.1. *Let $P(z)=z-kz^3+\dots$, $C(z)=2s$ be above and assume $k\neq 0$. Then*

$$S(P,C)=\frac{1}{\sqrt{k}}\mathbb{R}\subset\mathbb{C},$$

$$U(P,C)=\frac{\sqrt{-1}}{\sqrt{k}}\mathbb{R}\subset\mathbb{C},$$

where $\mathbb{R}\subset\mathbb{C}$ denotes the real number field.

In the following we analyze the germs $S(P,C)$ and $U(P,C)$.

First we suppose $k=1$, i.e. $P(z)=z-z^3+\dots$ and we analyze in the domain of convergence.

By the Taylor expansion

$$\frac{z}{\sqrt{1+2az^2}}=z-az^3+\frac{3}{2}a^2z^5-\frac{5}{2}a^3z^7+\frac{5\cdot 7}{2}a^4z^9-\dots,$$

we have

$$\frac{z}{\sqrt{1+2az^2}}<z-z^3+\dots<\frac{z}{\sqrt{1+2bz^2}},$$

for any sufficiently small real number $z>0$ and a, b with $0<b<1<a$. Define sequences of real numbers a_i, b_i and c_i by

$$a_{i+1}=\frac{a_i}{\sqrt{1+2aa_i^2}},\quad b_{i+1}=\frac{b_i}{\sqrt{1+2bb_i^2}},$$

$$c_{i+1}=P(c_i)=c_i-c_i^3+\dots,$$

with sufficiently small $a_0=b_0=c_0>0$. It is easy to see that

$$\frac{1}{a_i^2}=\frac{1}{a_0^2}+2ai,\quad \frac{1}{b_i^2}=\frac{1}{b_0^2}+2bi,$$

by which, with the inequality above, we have

$$\frac{1}{\sqrt{2ai+(1/a_0^2)}}=a_i\leqq c_i\leqq b_i=\frac{1}{\sqrt{2bi+(1/b_0^2)}}. \tag{1}$$

Next we claim

(2) Let $z_0 \in \mathbb{C} - 0$, $z_{i+1} = P(z_i)$ and suppose $z_i \to 0$. Then

$$|z_i|^2 > \frac{1}{2ai},$$

for $i = 0, 1, \ldots$ with some real number $a > 1$.

Proof. By the definition of z_i, $z_{i+1} = z_i - z_i^3 + O(z_i^4)$, we have

$$|z_{i+1}| \geqq |z_i| - |z_i|^3 - O'(|z_i|^4).$$

Applying (1), we have

$$|z_i| \geqq \frac{1}{\sqrt{2a'(i-i_0) + (1/|z_{i_0}|^2)}},$$

for an $a' > 1$ and sufficiently large i, from which we have (2).

Furthermore, under the same condition as (2), we claim

(3) $\arg z_i \to 0$ or π

To prove claim (3) we prove the following statements (4)–(6).

(4) If $\pi/3 < \arg z_i < 2\pi/3$ or $4\pi/3 < \arg z_i < 5\pi/3$ and $|z_i| \neq 0$ is sufficiently small (i is sufficiently large) then $|z_i| < |z_{i+1}|$.

Proof. By the equality $z_{i+1} = z_i - z_i^3 + O(z_i^4)$, we have

$$\begin{aligned}|z_{i+1}| &\geqq |z_i| + \cos|\arg z_i - \arg z_i^3| \cdot |z_i|^3 + O'(z_i^4)\\ &\geqq |z_i| + \cos \pi/3 \cdot |z_i|^3 + O'(|z_i|^4)\\ &\geqq |z_i| + \tfrac{1}{2}\cos \pi/3 \cdot |z_i|^3\\ &= |z_i| + \tfrac{1}{4}|z_i|^3\\ &> |z_i|,\end{aligned}$$

for sufficiently large i.

(5) If $0 < \theta < |\arg z_i| \leqq \pi/3$ and $|z_i| \neq 0$ is sufficiently small, then

$$|\arg z_{i+1}| < |\arg z_i| - \frac{1}{4\pi} \sin\theta \cdot |z_i|^2.$$

Proof. By the definition of z_i, we have

$$\begin{aligned}\log z_{i+1} &= \log(z_i - z_i^3 - O(z_i^4))\\ &= \log z_i + \log(1 - z_i^2 + O'(z_i^3))\\ &= \log z_i - z_i^2 + O''(z_i^3),\end{aligned}$$

from which we have

$$\begin{aligned}|\arg z_{i+1}| &= \left|\frac{1}{2\pi\sqrt{-1}} \operatorname{Im} \log z_{i+1}\right|\\ &= \left|\frac{1}{2\pi\sqrt{-1}} \operatorname{Im} \log z_i - \frac{1}{2\pi\sqrt{-1}} \operatorname{Im} z_i^2 + \frac{1}{2\pi\sqrt{-1}} \operatorname{Im} O''(z_i^3)\right|\\ &< \left|\arg z_i - \frac{1}{2\pi}|\sin\theta|\,|z_i|^2 + \frac{1}{2\pi\sqrt{-1}} \operatorname{Im} O''(z_i^3)\right|\\ &< |\arg z_i| - \frac{1}{4\pi} \sin\theta \cdot |z_i|^2\end{aligned}$$

for sufficiently small z_i, where we take the branch of $\log z$ such that $\log 1 = 0$.

Similarly to (5) above, we can prove

(6) If $0 < \theta < |\arg z_i - \pi| \leqq \pi/3$ and $z_i \neq 0$ is sufficiently small, then

$$|\arg z_{i+1} - \pi| < |\arg z_i - \pi| - \frac{1}{4\pi}|\sin\theta| \cdot |z_i|^2.$$

By (4) and (5), (6), we see that if $z_i \to 0$ then

$$0 \leqq |\arg z_i| < \frac{\pi}{3} \text{ or } 0 \leqq |\arg z_i - \pi| < \frac{\pi}{3},$$

for any sufficiently large i, and

$$\lim_{i\to\infty} \arg z_i \in \left(-\frac{\pi}{3}, \frac{\pi}{3}\right) \bigcup \left(\frac{2}{3}\pi, \frac{4}{3}\pi\right).$$

Now we prove claim (3). We suppose $\arg z_i \to \theta$, $0 < \theta < \pi/3$. Then by (5) and (2), we have

$$\begin{aligned}|\arg z_{i+1}| &< |\arg z_i| - \frac{1}{4\pi}\sin\theta \cdot |z_i|^2 \\ &< |\arg z_i| - \frac{1}{4\pi}\sin\theta \cdot \frac{1}{2ai}.\end{aligned}$$

Since

$$\sum_{k=i_0}^{i} \frac{1}{2ak} \to \infty \text{ as } i \to \infty,$$

it then follows that

$$|\arg z_i| \to \infty \text{ as } i \to \infty.$$

But this is a contradiction, so we have proven that $\theta = 0$.

Similarly we can prove that if $|\arg z_i| \to \theta$, $0 \leqq |\theta - \pi| < \pi/3$, then $\theta = \pi$. This completes the proof of claim (3).

(It is known that a function germ $x + ax^i + \ldots$ on $\mathbb{C}$ at 0 is topologically conjugate with the normal form $x + x^i$ by Camacho.)

Proof of Proposition 1.4.1. First we assume that $k = 1$. By claim (3)

$$\arg P^i(z) \to 0, \pi,$$

if $P^i(z) \to 0$. Since the expansion C preserves $\arg P^i(z)$, we see that $S(P, C) \subset \mathbb{R} \subset \mathbb{C}$. To see that $S(P, C) = \mathbb{R} \subset \mathbb{C}$ is an easy exercise with the order of the convergence in (1).

Using the coordinate $z' = \sqrt{-1}\, z$, P and C are of the forms:

$$\begin{aligned}P^{-1}(z') &= z' + z'^3 + \ldots \\ &= \sqrt{-1}\,(z - z^3 + \ldots) = \sqrt{-1}\; P'(z) \\ C(z') &= \sqrt{-1}\; C(z).\end{aligned}$$

By this and the statement for $k = 1$, we have

$$U(P, C) = S(P^{-1}, C) = \sqrt{-1}\; S(P', C) = \sqrt{-1}\,\mathbb{R} \subset \mathbb{C}.$$

For $k \neq 1, 0$, by the linear coordinate change $h\colon \mathbb{C} \to \mathbb{C}$, $h(z) = \sqrt{k}z$, we can normalize $k = 1$.

i.e. $h \circ P \circ h^{-1}(z) = z - z^3 + \dots$. Then by the statement above for $k=1$, we have

$$\begin{aligned} h(S(P,C)) &= S(h \circ P \circ h^{-1}, h \circ C \circ h^{-1}) \\ &= S(h \circ P \circ h^{-1}, C) \\ &= \mathbb{R} \subset \mathbb{C}, \end{aligned}$$

and similarly

$$h(U(P,C)) = \sqrt{-1}\,\mathbb{R},$$

from which we have

$$S(P,C) = \frac{1}{\sqrt{k}}\mathbb{R}, \quad U(P,C) = \frac{\sqrt{-1}}{\sqrt{k}}\mathbb{R}.$$

This completes the proof of Proposition 1.4.1.

5. *Proof of Theorem* 1.5.1.

In this section, we prove the theorem:

THEOREM 1.5.1. *Let* $\mathscr{W} = (\mathscr{F}_1, \dots, \mathscr{F}_{n+1})$, $\mathscr{W}' = (\mathscr{F}'_1, \dots, \mathscr{F}'_{n+1})$ *be germs of non-degenerate analytic* $(n+1)$*-webs of* $\mathbb{C}^n$ *at* 0 *of codimension* 1, *and assume that for any* $i = 1, \dots, n+1$, *there are* $j, k \neq i$ *such that the restriction of the subwebs* $(\mathscr{F}_i, \mathscr{F}_j, \mathscr{F}_k)$, $(\mathscr{F}'_i, \mathscr{F}'_j, \mathscr{F}'_k)$ *to the intersections of leaves* $\bigcap_{l \neq i,j,k} L^l_0$, $\bigcap_{l \neq i,j,k} L'^l_0$, $0 \in L^l_0$, L'^l_0, $L^l_0 \in \mathscr{F}_l$, $L'^l \in \mathscr{F}'_l$ *are non-hexagonal. Let h be a germ of homeomorphism of* $(\mathbb{C}^n, 0)$ *such that*

$$h(\mathscr{W}) = \mathscr{W}', \text{ i.e. } h(\mathscr{F}_i) = \mathscr{F}'_i,\ i = 1, \dots, n+1.$$

Then h or the complex conjugate $\bar{h}$ *is a complex analytic diffeomorphism of* $(\mathbb{C}^n, 0)$.

Remark. The condition for $\mathscr{W}, \mathscr{W}'$ in the theorem is too strong. This is used only for the reduction to the case $n=2$.

Proof of Theorem 1.5.1. The statement is for germs of mappings and subsets at the origin in $\mathbb{C}^n$. But throughout the proof, we suppose all mappings and subsets are given by their representatives in an open neighborhood of the origin, and we shall analyze the germs by those representatives. For simplicity we sometimes denote the germs ambiguously as subsets and mappings of $\mathbb{C}^n$ when this notation causes no confusion.

Reduction to the case $n=2$. Let u_l, u'_l be analytic level functions for $\mathscr{F}_l$, $\mathscr{F}'_l$. Then $h\left(\bigcap_{l \neq i,j,k} L^l_0\right) = \bigcap_{l \neq i,j,k} L'^l_0$ and h is a homeomorphism of non-degenerate 3-webs $(\mathscr{F}_i, \mathscr{F}_j, \mathscr{F}_k)$, $(\mathscr{F}'_i, \mathscr{F}'_j, \mathscr{F}'_k)$ of the intersections. Applying the statement for $n=2$ to these 3-webs, we see that the restriction of h (or $\bar{h}$) to $\bigcap_{l \neq i,j,k} L^l_0$ is an analytic diffeomorphism and in particular this induces diffeomorphisms h_l of $(\mathbb{C}, 0)$ so that the level functions u_l, u'_l are conjugate:

$$\begin{array}{ccc} u_l\colon \left(\bigcap_{l \neq i,j,k} L^l_0, 0\right) & \longrightarrow & (\mathbb{C}, 0) \\ \downarrow & & \downarrow h_l \\ u'_l\colon \left(\bigcap_{l \neq i,j,k} L'^l_0\right) & \longrightarrow & (\mathbb{C}, 0) \end{array} \quad \text{commutes,}$$

and either h_l, $l = i,j,k$ are analytic or $\bar{h}_l$, $l = i,j,k$ are analytic.

Since h maps a leaf of $\mathscr{F}_l$ to a leaf of $\mathscr{F}'_l$ for $l=1,\ldots,n+1$, the level functions u_l and u'_l are conjugate by h and h_l:

$$\begin{array}{ccc} u_l\colon(\mathbb{C}^n,0) & \longrightarrow & (\mathbb{C},0) \\ {\scriptstyle h}\big\downarrow & & \big\downarrow{\scriptstyle h_l} \\ u'_l\colon(\mathbb{C}^n,0) & \longrightarrow & (\mathbb{C},0) \end{array}$$

commutes for $l=1,\ldots,n+1$. The result that h_i, h_j, h_k or their conjugate are analytic holds for any choice of i,j,k. Therefore h_i or $\bar{h}_i$ are uniformly analytic. By the diagram (*), we have

$$h=(u'_1,\ldots,u'_n)^{-1}(h_1,\ldots,h_n)(u_1,\ldots,u_n),$$

so h or $\bar{h}$ is analytic. This proves that the case $n=2$ implies the case $n\geqslant 3$.

Proof for the case $n=2$. First we suppose that the web $\mathscr{W}, \mathscr{W}'$ are of the normal form defined by the following level functions with web functions:

(a)
$$u_1=u'_1=x,\ u_2=u'_2=y,$$
$$u_3=\omega(x,y)=x+y+k(x^2y-xy^2)+R_4(x,y),$$
$$u'_3=\omega'(x,y)=x+y+k'(x^2y-xy^2)+R'_4(x,y),$$

and

$$W(t_1,t_2,t_3)=\omega(t_1,t_2)-t_3,$$
$$W'(t_1,t_2,t_3)=\omega'(t_1,t_2)-t_3,$$

where R_4, R'_4 are the remainder terms of order $\geqq 4$ such that $R_4(t,0)=R'_4(t,0)=0$, $R_4(0,t)=R'_4(0,t)=0$ and $R_4(t,t)=R'_4(t,t)=0$. Then the leaves are $L^1_0=L'^1_0=y$-axis and $L^2_0=L'^2_0=x$-axis in $\mathbb{C}^2$.

We introduce two invariant germs of subsets associated with the web $\mathscr{W}$:

$$S^i_p(\mathscr{W})=S(((u_{3-i}|L^i_0)\circ P^i_{0,p}(\mathscr{W})\circ(u_{3-i}|L^i_0)^{-1})^2,(u_{3-i}|L^i_0)\circ C^{3,i}_0(\mathscr{W})\circ(u_{3-i}|L^i_0)^{-1})$$
$$=S((\bar{P}^{3-i}_p(\mathscr{W}))^2,C)$$

and

$$U^i_p(\mathscr{W})=U((P^{3-i}_p(\mathscr{W}))^2,C),$$

for $p\in L^i_0$, $i=1,2$, where S, U are the stable and unstable sets, $P^i_{0,p}(\mathscr{W})$, $\bar{P}^{3-i}_p(\mathscr{W})$, $C^{3,i}_0(\mathscr{W})$ are the mappings associated with the web $\mathscr{W}$ and $C(z)=2z$. [By the form of $\mathscr{W}$ $C^3_0(z)=2z$, so $C^{3,1}_0(\mathscr{W})$ $(0,z)=(0,2z)$ and $C^{3,1}_0(\mathscr{W})$ $(z,0)=(2z,0)$. For definitions, see §1.2.] Note that $S^1_0(\mathscr{W})=S^2_0(\mathscr{W})$ and $U^1_0(\mathscr{W})=U^2_0(\mathscr{W})$ holds by definition.

First we assume the following condition (G):

(G) $k,k'\neq 0$ and the function $k(W)\cdot W_2(x,0,x)$ restricted to $L^2_0=x$-axis is non-singular at 0.

Then arg $k(W)\cdot W_2(x,0,x)$ is non-singular at 0 restricted to the real lines S^2_0 or $U^2_0=\sqrt{-1}\,S^2_0$ as a real-valued analytic function. Here we assume the former case (for the latter case, the argument goes the same).

By Proposition 1.2.1, we have

$$(\bar{P}^2_{(x,0)})^2(t)=t-k(W)(x,0)\cdot W^2_2(x,0,x)\cdot t^3+R_4(x,0)\ (t),$$

where $k(W)$ is the web curvature of u_i, W, $k(W)$ $(0,0)\cdot W_2(0,0,0)=k(W)$ $(0,0)=k$ and $W_i=\partial W/\partial t_i$. By Proposition 1.4.1, we have

$$S^1_{(x,0)}=\frac{1}{\sqrt{k(W)(x,0)\cdot W^2_2(x,0,x)}}\,\mathbb{R}\subset\mathbb{C},$$

$$U^1_{(x,0)}=\frac{\sqrt{-1}}{\sqrt{k(W)(x,0)\cdot W^2_2(x,0,x)}}\,\mathbb{R}\subset\mathbb{C},$$

and similarly we have

$$S^2_{(0,y)} = \frac{1}{\sqrt{k(W)(0,y)\cdot W^2_1(0,y,y)}}\,\mathbb{R} \subset \mathbb{C},$$

$$U^2_{(0,y)} = \frac{\sqrt{-1}}{\sqrt{k(W)(0,y)\cdot W^2_1(0,y,y)}}\,\mathbb{R} \subset \mathbb{C},$$

Define the following real analytic mappings.

$$M_1\colon (\mathbb{R}\times S^2_0,(0,0)) \to (\mathbb{C},0),$$
$$M_2\colon (S^1_0\times U^1_0,(0,0)) \to (\mathbb{C},0),$$
$$M_3\colon (U^1_0\times S^1_0,(0,0)) \to (\mathbb{C},0),$$

by

$$M_1(\lambda,x) = \frac{\lambda}{\sqrt{2k(W)(x,0)\,W^2_2(x,0,x)}},\quad \lambda\in\mathbb{R},\quad x\in S^2_0,$$
$$M_2(x,y) = x +^{2,3}_0 y,\quad (x,0)\in S^1_0,\quad y\in U^1_0,$$
$$M_3(x,y) = x +^{2,3}_0 y,\quad (x,0)\in U^1_0,\quad y\in S^1_0,$$

where $+^{2,3}_0$ is the local translation map of the range of u_2 defined in Section 1. Denote that

$$C^1_x = M_1(\mathbb{R},x),\quad C^2_y = M_2(S^1_0,y),\quad C^3_y = M_3(U^1_0,y)$$

and let G_1, G_2, G_3 be the collections of manifolds

$$G_1 = \{C^1_x \mid x\in S^2_0\},\quad G_2 = \{C^2_y \mid y\in U^1_0\},\quad G_3 = \{C^3_y \mid y\in S^1_0\}.$$

PROPOSITION 1.5.2. *Assume that the real-valued function* arg $k(W)$ $(x,0)\cdot W^2_2(x,0,x)$ *restricted to the real line* $S^2_0(W)$ *is topologically non-singular at* $0\in\mathbb{C}^2$. *Then* G_2, G_3 *are germs of real analytic foliations of* $\mathbb{C}=\mathbb{R}^2$ *of codimension* 1, *and* G_1 *forms a real analytic foliation of codimension* 1 *on a germ of a deleted neighborhood* U *of* S^1_0-0 *in* $\mathbb{C}$ *at the origin* $0\in\mathbb{C}^2$, *on which* (G_1,G_2,G_3) *forms a non-degenerate* 3*-web. Here a germ of a deleted neighborhood means a germ of a subset at* $0\in\mathbb{C}^2$ *represented by a set of the form* $U'-(S^1_0-0)$, U' *being an open neighborhood of* S^1_0-0 *in* $\mathbb{C}$ *at the origin.*

Proof. Since S^1_0, $U^1_0=\sqrt{-1}\;S^1_0$ are real lines and $dM_i(0)=\mathrm{id}\colon T_0\mathbb{C}\to T_0\mathbb{C}$ for $i=2,3$, M_2, M_3 are germs of real analytic diffeomorphisms and G_2, G_3 are germs of non-singular real analytic foliations of codimension 1. Clearly G_1 and G_3 are in general positions. So we consider for G_1 and G_2.

Since G_2 is real analytic, the singular point set

$$\Sigma = \{(\lambda,x)\in\mathbb{R}\times S^2_0 \mid \text{the leaf } C^1_x \text{ is not transversal to the foliation } G^2_2 \text{ at } M_1(\lambda,x)\}$$

is real analytic. It is easy to see that if $\Sigma=\mathbb{R}\times S^2_0$ then $C^1_x=S^1_0$ for any $x\in S^2_0$. However $\arg k(W)$ $(x,0)\cdot W^2_2(x,0,x)$ is topologically non- singular at $0\times S^2_0$, so M_1 is an open map outside the subset $0\times S^2_0$ by the form of M_1. Therefore Σ is a proper real analytic subset, and there is a germ of deleted neighborhood U of $(\mathbb{R}-0)\times 0$ in $(\mathbb{R}-0)\times S^2_0-\Sigma$ at 0×0 and the foliations G_1, G_2 are in general positions on the germ of deleted neighborhood $M_1(U)$ of S^1_0-0 in $\mathbb{C}$ at 0. This proves Proposition 1.5.2.

Now we prove Theorem 1.5.1. The following is a part of the theorem.

PROPOSITION 1.5.3. *Assume condition* (G) *and the other assumptions above. Let* h, h_i *be germs of homeomorphisms such that the following diagram commutes*:

$$(**)\qquad \begin{array}{ccc} u_i\colon (\mathbb{C}^2,0) & \longrightarrow & (\mathbb{C}^1,0) \\ h\downarrow & & \downarrow h_i \\ u_i'\colon (\mathbb{C}^2,0) & \longrightarrow & (\mathbb{C}^1,0), \end{array}$$

for $i=1,2,3$. *Then* $h_1=h_2=h_3$ *and* $h=(h_1,h_2)$ *and* h, h_i *or their conjugates* $\bar{h}$, $\bar{h}_i$ *are complex analytic diffeomorphisms.*

Proof. It is clear that $h_1=h_2=h_3$ and $h=(h_1,h_2)$ hold by the normal form (a). So we have only to prove h_2 is a complex analytic diffeomorphism at $0\in\mathbb{C}$.

Recall that the real analytic 3-webs $G=(G_1,G_2,G_3)$, $G'=(G_1',G_2',G_3')$ of codimension 1 of $\mathbb{C}$ are constructed purely by the topological structure of the webs $\mathscr{W}$, $\mathscr{W}'$. So we have

$$h_2(G_i)=G_i',$$

for $i=1,2,3$, and in particular

$$h_2(S^1_{(x,0)}(W))=S^1_{(h_1(x),0)}(W'),$$
$$h_2(U^1_{(x,0)}(W))=U^1_{(h_1(x),0)}(W'),$$

and

$$h_1(S_0^2(W))=S_0^2(W'),$$

from which, with Proposition 1.4.1, we have

$$h_2\left(\frac{\mathbb{R}}{\sqrt{k(W)(x,0)\cdot W_2'^2(x,0,x)}}\right)=\frac{\mathbb{R}}{\sqrt{k(W')(h_1(x),0)\cdot W_2'^2(h_1(x),0,h_1(x))}}$$
$$h_2\left(\frac{\sqrt{-1}\,\mathbb{R}}{\sqrt{k(W)(x,0)\cdot W_2'^2(x,0,x)}}\right)=\frac{\sqrt{-1}\,\mathbb{R}}{\sqrt{k(W')(h_1(x),0)\cdot W_2'^2(h_1(x),0,h_1(x))}}$$

for $x\in S_0^2(W)\subset\mathbb{C}$. Since the real-valued function $\arg k(W)(x,0)\cdot W_2'^2(x,0,x)$ restricted to the real line $S_0^2(W)\subset\mathbb{C}$ is non-singular at $0\in\mathbb{C}$, the function $\arg k(W')(x,0)\cdot W_2'^2(x,0,x)$ restricted to $S_0^2(W')\subset\mathbb{C}$ is also topologically non-singular at $0\in\mathbb{C}$. So, by Proposition 1.5.2, (G_1,G_2,G_3), (G_1',G_2',G_3') form non-degenerate real analytic 3-webs of codimension 1 on germs of deleted neighborhoods U, U' of $S_0^1(W)-0$, $S_0^1(W')-0$ in $\mathbb{C}$ at the origin. By Dufour's theorem (Lemma 1.1.1), h_2 is a real analytic diffeomorphism restricted to the non-empty set $U\cap h_2^{-1}(U')$.

By the diagram (**), we have the following commutative diagram:

$$(***)\qquad \begin{array}{ccc} +_0^{2,3}(W)x\colon\ (\mathbb{C},0) & \longrightarrow & (\mathbb{C},x) \\ h_2\downarrow & & \downarrow h_2 \\ +_0^{2,3}(W')h_2(x)\colon\ (\mathbb{C},0) & \longrightarrow & (\mathbb{C},h_2(x)), \end{array}$$

for any point $x\in\mathbb{C}$. By this diagram, we see that

h_2 *is a real analytic diffeomorphism of* $(\mathbb{C},0)$.

Since the homeomorphism h_2 carries all right-angles in $\mathbb{C}$ formed by real lines $S^1_{(x,0)}(W)$ and $U^1_{(x,0)}(W)=\sqrt{-1}\,S^1_{(x,0)}(W)$ passing through $0\in\mathbb{C}$ for $x\in S_0^2$ to the right-angles of $S^1_{(h_1(x),0)}(W')$ and $U^1_{(h_1(x),0)}(W')$ at $0\in\mathbb{C}$, we see that h_2 or the complex conjugate $\bar{h}_2$ is conformal at $0\in\mathbb{C}$, respectively whether $\bar{h}_2$ is orientation preserving or not. Again by the diagram (***), h_2 or $\bar{h}_2$ is conformal on a neighborhood of $0 \in \mathbb{C}$. Then Riemann's theorem says that h_2 or $\bar{h}_2$ is complex analytic at $0 \in \mathbb{C}$.

This completes the proof of Proposition 1.5.3, which is a particular case of Theorem 1.5.1 for $n=2$ and condition (G) holding. Next we consider for general case of $n=2$.

Since the curvatures $k(W)$, $k(W')$ of $\mathscr{W}, \mathscr{W}'$ are not identically zero on a neighborhood of the origin $0 \in \mathbb{C}^2$ by Theorem 1.1.1 and the assumption of Theorem 1.5.1, there is a point $p \in \mathbb{C}^2$ sufficiently close to the origin such that $k(W)(p), k(W')(h(p)) \neq 0$. By Proposition 1.3.2, $k(W) \cdot W_i^2(u_1, u_2, u_3)$ is not constant on a leaf $L_q^i = u_i^{-1}(u_i(q))$ for a sufficiently small $u_i(q)$ and an $i = 1, 2, 3$. So we may assume, in addition, that $k(W) \cdot W_2^2(u_1, u_2, u_3)$ is non-singular at p restricted to L_p^i. This property is inherited after changing the functions u_i to the normal form (a). Applying Proposition 1.5.3 to the re-formed webs, we see that $h(\bar{h})$ is complex analytic at $p \in \mathbb{C}^2$ and $h_j(\bar{h}_j)$ is complex analytic at $u_j(p) \in \mathbb{C}$ for $j = 1, 2, 3$. Again by the diagram (***):

$$(***) \qquad \begin{array}{rcl} +\int_0^{j,k} u_i(p)\colon & (\mathbb{C}, 0) \longrightarrow & (\mathbb{C}, u_j(p)) \\ & h_j \big\downarrow \qquad & \big\downarrow h_j \\ +\int_0^{j,k} u_i'(h(p))\colon & (\mathbb{C}, 0) \longrightarrow & (\mathbb{C}, u_j'(h(p))), \end{array}$$

[which we gave in the proof of Proposition (1.5.3)], we see that $h_j(\bar{h}_j)$, $j = 1, 2, 3$ are complex analytic at $0 \in \mathbb{C}$. Hence $h = (u_1, u_2)^{-1}(h_1, h_2)(u_1, u_2)$ (or $\bar{h}$) is complex analytic at the origin $0 \in \mathbb{C}^2$.

This completes the proof of Theorem 1.5.1.

§2. APPLICATION TO THE PROJECTIVE GEOMETRY OF PROJECTIVE SPACE CURVES

In this section we study the geometric structure of the *d-webs $\mathscr{W}_C$ generated by* projective curves $C^d \subset \mathbb{P}_n$ of degree d. The d-web $\mathscr{W}_C$ is defined to be the collection of dual hyperplanes $y^\vee = \mathbb{P}_{n-1} \subset \mathbb{P}_n^\vee$, $y \in C$. Let $\Sigma(\mathscr{W}_C) \subset \mathbb{P}_n^\vee$ be the set of points $x \in \mathbb{P}_n^\vee$ where $\mathscr{W}_C$ is not a non-singular d-web (for definition, see §2.1). The structure of $\Sigma(\mathscr{W}_C)$ is studied in the later sections.

2.1. *Proof of Theorem* 2.1.3

We say two webs $\mathscr{W}_C$, $\mathscr{W}_{C'}$ generated by algebraic curves $C, C' \subset \mathbb{P}_n$ are *topologically equivalent* if there is a homeomorphism h of $\mathbb{P}_n$ such that, for any leaf $x^\vee$, $x \in C$, of $\mathscr{W}_C$, the image $h(x^\vee)$ is a leaf $x'^\vee$ for an $x' \in C'$. Then we denote $h\colon \mathscr{W}_C \to \mathscr{W}_{C'}$ or $h(\mathscr{W}_C) = \mathscr{W}_{C'}$. Our problem is to classify all webs $\mathscr{W}_C$ up to this equivalence relation.

In this section, we denote by $P(x_1, \ldots, x_a)$ the subspace spanned by $x_1, \ldots, x_a$ in $\mathbb{P}_n$ and denote by $\mathscr{W}_C(x_1, \ldots, x_a)$ the restriction of $\mathscr{W}_C$ to $P(x_1, \ldots, x_a)$.

We define two singular sets $\mathrm{envl}(\mathscr{W}_C)$ and $\mathrm{degn}(\mathscr{W}_C)$ as follows:

$$\begin{aligned} \mathrm{envl}(\mathscr{W}_C) &= \{x \in \mathbb{P}_n^\vee \mid x^\vee \text{ has contact with } C \text{ at a smooth point or } x^\vee \cap \Sigma(C) \neq \emptyset\} \\ &= \{x \in \mathbb{P}_n^\vee \mid m_{x'}(x^\vee, C) \geqq 2 \text{ for an } x' \in x^\vee\} \\ &= \{x \in \mathbb{P}_n^\vee \mid \text{ the geometric number of points of } x^\vee \cap C \text{ is less than } d\} \end{aligned}$$

$$\mathrm{degn}(\mathscr{W}_C) = \overline{\{x \in \mathbb{P}_n^\vee \mid x^\vee \cap C \text{ is non-singular and degenerate in } x^\vee = \mathbb{P}_{n-1}\}},$$

where $\Sigma(C)$ is the set of singular points of C and "degenerate" means that some distinct n points $x_1, \ldots, x_n \in x^\vee \cap C$ are coplanar in $x^\vee$, i.e. $x_1, \ldots, x_n$ do not span $x^\vee$. The variety envl $(\mathscr{W}_C)$ is known as the *dual variety* of C defined similarly to the dual plane curve (see [22, 28]).

The detailed structure of $\Sigma(\mathscr{W}_C^\vee)$ is investigated in §2.2. First we offer the following proposition.

PROPOSITION 2.1.1. $\Sigma(\mathscr{W}_C^\vee)=\operatorname{degn}(\mathscr{W}_C^\vee)\cup\operatorname{envl}(\mathscr{W}_C^\vee)$.

Proof. Let the multiplicity be $m_{x_i}(x^\vee, C)=m_i$ for $x_i\in x^\vee\cap C$. Then the geometric number of points of the intersection $x^\vee\cap C$ is $d'=d-\Sigma(m_i-1)$. This shows that just d' leaves of $\mathscr{W}_C^\vee$ are passing through x. So we have $\operatorname{envl}(\mathscr{W}_C^\vee)\subset\Sigma(\mathscr{W}_C^\vee)$.

Let $x\notin\operatorname{envl}(\mathscr{W}_C^\vee)$. Then $m_i=1$ for any $x_i\in x^\vee\cap C$ and $x^\vee$ meets C transversely at d distinct points $x_1,\dots,x_d$. The germs (C,x_i) generate germs of non-singular foliations $\mathscr{F}_i$ at x, which form a non-degenerate d-web of codimension 1 if and only if $x^\vee\cap C$ is nondegenerate in $x^\vee=\mathbb{P}_{n-1}$.

PROPOSITION 2.1.2. *Let $C, C'\subset\mathbb{P}_n$ be projective curves and h a homeomorphism of the dual space $\mathbb{P}_n^\vee$ such that $h(\mathscr{W}_C^\vee)=\mathscr{W}_{C'}^\vee$. Then h induces a homeomorphism $h^\vee\colon C\to C'$ by $h(x^\vee)=h^\vee(x)^\vee$ for $x\in C$, which possesses the properties $h(P(x_1,\dots,x_{n-2})^\vee)=P(h(x_1),\dots,h(x_{n-2}))^\vee$ and $h(\mathscr{W}_C^\vee(x_1,\dots,x_{n-2}))=\mathscr{W}_{C'}^\vee(h^\vee(x_1),\dots,h^\vee(x_{n-2}))$.*

Proof. Clearly $h^\vee$ is a continuous map of C into C', and $(h^{-1})^\vee\circ h^\vee=\mathrm{id}$ holds by definition. So $h^\vee$ is a homeomorphism. Since $P(x_1,\dots,x_{n-2})^\vee\cap x^\vee=P(x_1,\dots,x_{n-2},x)^\vee$ for $x\in C$, and $h(P(x_1,\dots,x_{n-2},x)^\vee)=P(h^\vee(x_1),\dots,h^\vee(x_{n-2}),h^\vee(x))$, we have $h(\mathscr{W}_C^\vee(x_1,\dots,x_{n-2}))=\mathscr{W}_{C'}^\vee(h^\vee(x_1),\dots,h^\vee(x_{n-2}))$.

Now we state our main theorem in this section.

THEOREM 2.1.3. *Let $C, C'\subset\mathbb{P}_n$ be algebraic curves in the projective n-space $(n\geqq 2)$ of degree d, and $\mathscr{W}_C^\vee, \mathscr{W}_{C'}^\vee$ be the d-webs generated by C, C', respectively, and let h be a homeomorphism of the dual space $\mathbb{P}_n^\vee$ such that $h(\mathscr{W}_C^\vee)=\mathscr{W}_{C'}^\vee$. If C, C' are irreducible and non-degenerate,* i.e. *are not contained in a hyperplane, and $d\geqq n+2$, then h or the complex conjugate $\bar{h}$ is a projective linear transformation of $\mathbb{P}_n^\vee$ and in particular C' is isomorphic to either C or its conjugate $\bar{C}$: the induced homeomorphism $h^\vee\colon C\to C'$* is given by $({}^th)^{-1}\colon C\to C'$ *or* $\overline{({}^th)}^{-1}\colon C\to\bar{C}'\to C'$.

Proof. Let $y\in\mathbb{P}_n^\vee-\Sigma(\mathscr{W}_C^\vee)$ and $\{x_1,\dots,x_d\}=y^\vee\cap C$ and $\pi\colon\mathbb{P}_n\to\mathbb{P}_2$ be the projection with the center $\mathbb{P}_{n-3}=P(x_1,\dots,x_{n-2})$ to $P(x_1,\dots,x_{n-2})^{\vee *}$, where * denotes the dual projective space of itself as $\mathbb{P}_2$ not in $\mathbb{P}_n$. The closure of the image $\pi(C-x_1,\dots,x_{n-2})\subset\mathbb{P}_2$ is again an irreducible and non-degenerate algebraic curve of degree $d-(n-2)$, which we denote by $C(x_1,\dots,x_{n-2})$. Since $d\geqq n+2$, we have $d-(n-2)\geqq 4$.

Now we prove

PROPOSITION 2.1.4. *The restriction $\mathscr{W}_C^\vee(x_1,\dots,x_{n-2})$ is the web of $\mathbb{P}_2=P(x_1,\dots,x_{n-2})^\vee$ generated by the algebraic curve $C(x_1,\dots,x_{n-2})\subset\mathbb{P}_2=P(x_1,\dots,x_{n-2})^{\vee *}$ (dual space).*

Proof. The leaves $P(x_1,\dots,x_{n-2})^\vee\cap x^\vee=P(x_1,\dots,x_{n-2},x)^\vee$, $x\in C$, of $\mathscr{W}_C^\vee(x_1,\dots,x_{n-2})$ are the intersections of the dual hyperplane of $\pi(x)$ in $\mathbb{P}_n$ with $P(x_1,\dots,x_{n-2})$. So we see that $\mathscr{W}_C^\vee(x_1,\dots,x_{n-2})=\mathscr{W}_{C(x_1,\dots,x_{n-2})}$. This proves the proposition.

By the above proposition, Graf–Sauer's theorem applies to the algebraic plane curve $C(x_1,\dots,x_{n-2})$ and the induced $d-(n-2)$-web of the intersection $\mathbb{P}_2=P(x_1,\dots,x_{n-2})\subset\mathbb{P}_n^\vee$ of leaves $x_i^\vee$, and says that any 3-subweb of $\mathscr{W}_C^\vee(x_1,\dots,x_{n-2})$ is nowhere hexagonal outside the singular set $\Sigma(\mathscr{W}_C^\vee(x_1,\dots,x_{n-2}))$.

Since $\Sigma(\mathscr{W}_C) = \mathrm{degn}(\mathscr{W}_C) \cup \mathrm{envl}(\mathscr{W}_C)$ is defined pure topologically, $h(\Sigma(\mathscr{W}_C)) = \Sigma\mathscr{W}_{C'}$ holds.

Theorem 1.5.1 says that h or the conjugate $\bar{h}$ is complex analytic outside the singular set $\Sigma(\mathscr{W}_C)$, which is proper subvariety of $\mathbb{P}_n^\vee$ since C is non-degenerate. By Hartog's extension theorem, h or $\bar{h}$ is a complex analytic automorphism of $\mathbb{P}_n^\vee$, hence a projective linear transformation of $\mathbb{P}_n^\vee$.

The other statement is easy to see. This completes the proof of Theorem 2.1.4.

In terms of algebraic geometry, the theorem can be rephrased as follows.

COROLLARY 2.1.5. *Let C, C' be Riemann surfaces and let E, E' be linear systems of effective divisors of degree d with no base points such that the associated morphisms $\mathscr{I}_E$: $C \to E^\vee$, $\mathscr{I}_{E'}$: $C' \to E'^\vee$ are birational and $d-2 \geqq \dim E = \dim E' \geqq 2$. Suppose that there is a homeomorphism h: $C \to C'$ such that $h(E) = E'$, i.e. $h(\Sigma a_i x_i) = \Sigma a_i h(x_i) \in E' \Leftrightarrow a_i x_i \in E$. Then h: $C \to C'$ is a holomorphic or anti-holomorphic diffeomorphism respectively whether h is orientation preserving or not.*

Proof. We identify the linear system $|D|$, $D \in E$ with $\mathbb{P}(H^0(C, O(|D|)))$ by $\Sigma a_i x_i \in |D| \leftrightarrow s \in H^0(C, O(|D|))$ with $s^{-1}(0) = \Sigma a_i x_i$, and we suppose $E \subset |D| = \mathbb{P}_{\dim |D|}$. The morphism $\mathscr{I}_E$: $C \to E^\vee$ is defined by $x \in C \to H_x^\vee$, where $H_x = \{\Sigma a_i x_i \in E | x_i = x \text{ for an } i\} \subset E$. Then the image $\tilde{C} \subset E^\vee$ of C is a non-degenerate curve of degree d which generates the d-web $\mathscr{W}_{\tilde{C}}$ on E with the leaves H_x, $x \in C$.

The homeomorphism h: $C \to C'$ preserves the linear systems E, E' so h induces a homeomorphism $h^\vee$: $E \to E'$ which maps a leaf H_x, $x \in C$, to a leaf $H_{h(x)}$, $h(x) \in C'$, therefore $h^\vee(\mathscr{W}_{\tilde{C}}) = \mathscr{W}_{\tilde{C}'}$. Then Theorem 2.1.3 says $h^\vee$ or the complex conjugate $\bar{h}^\vee$ is a projective linear transformation and $({}^t h^\vee)^{-1}$ or $({}^t \bar{h}^\vee)^{-1}$ is a transformation of $\tilde{C}$ to $\tilde{C}'$, which lifts to an isomorphism of C to C', i.e. the original homeomorphism h.

This completes the corollary.

Riemann–Roch theorem says that $\dim |D| = d + 1 - g + \dim |K - D|$, where K is the canonical divisor and g is the genus of C. So, if $d \geqq g + 2$ then $\dim |D| \geqq 2$ and a linear system E of dimension 2 (net) exists. So roughly speaking, a complex structure of a Riemann surface C of genus g is determined by a 2-dimensional family of linearly equivalent $g+2$ ($g \geqq 2$) or 4 ($g = 1$) point subsets of C.

2.2. *Structure of the envelope set envl $(\mathscr{W}_C)$: the dual space curves and the dual webs*

In this section, we turn to the study of the envelope set $\mathrm{envl}(\mathscr{W}_C) \subset \mathbb{P}_n^\vee$ of the web $\mathscr{W}_C$ generated by a projective curve $C \subset \mathbb{P}_n$.

Let ϕ: $\tilde{C} \to C$ be the normalization and $\tilde{\phi} = (\phi_0, \ldots, \phi_n)$: $\tilde{C} \to \mathbb{C}^{n+1} - 0$ be a local lift of ϕ. Suppose that $\tilde{\phi}$ is non-degenerate, i.e. the Wronskian $W(\phi_0, \ldots, \phi_n)$ is not identically zero on $\tilde{C}$, in other words, C is not contained in a hyperplane. Let $C_{\mathrm{reg}} = C - \mathrm{sing}\, C$, $C_0 = C_{\mathrm{reg}} - \phi(W^{-1}(0))$ and $\tilde{C}_0 = \tilde{C} - W^{-1}(0)$.

Let $\Sigma^{1,\overset{i}{\ldots},1}(\mathscr{W}_C) \subset \mathrm{envl}(\mathscr{W}_C) \subset \mathbb{P}_n^\vee$ be the set of points y whose dual hyperplanes $y^\vee$ have contact with C_0 of order $\geqq i+1$, and $\mathrm{envl}^i(\mathscr{W}_C) = \bar{\Sigma}^{1,\overset{i}{\ldots},1}(\mathscr{W}_C)$ (closure). The *osculating i-plane* $\mathrm{Osc}^i\, C_{\phi(t)}$ of C at $\phi(t) \in C_0$ is the i-plane $\mathbb{P}_i$ which has a contact with C at $\phi(t)$ of order $\geqq i+1$. It is the i-space spanned by the vectors $\phi(x), \ldots, \phi^{(i-1)}(t)$ if rank

$$\begin{pmatrix} \phi_0, \ldots, \phi_n \\ \phi_0^{i-1}, \ldots, \phi_n^{i-1} \end{pmatrix}(t) = i \text{ or especially } W(\phi_0, \ldots, \phi_n)(t) \neq 0. \text{ The osculating } i\text{-planes}$$

give the i-bundle. $\mathrm{Osc}^i\, C \to \tilde{C}$ over $\tilde{C}$, which we call the *osculating i-bundle* of C, and we denote the restriction over $\tilde{C}_0$ by $\mathrm{Osc}^i\, C_0$.

By definition, $\Sigma^{1,\overset{i}{\cdots},1}(\mathscr{W}_C)$, $\mathrm{envl}^i(\mathscr{W}_C)$ are the union of the dual spaces of $\mathrm{Osc}^i C_x$, $x \in C_0$, C, respectively, and we have

$$\mathbb{P}_n^\vee = \Sigma^0(\mathscr{W}_C) \supset \Sigma^1(\mathscr{W}_C) \supset \dots \supset \Sigma^{1,\overset{n-1}{\cdots},1}(\mathscr{W}_C) \supset \dots,$$

$$\mathbb{P}_n^\vee = \mathrm{envl}^0(\mathscr{W}_C) \supset \mathrm{envl}^1(\mathscr{W}_C) \supset \dots \supset \mathrm{envl}^{n-1}(\mathscr{W}_C) \supset \dots.$$

[In the following, we show that $\dim \Sigma^{1,\overset{i}{\cdots},1}(\mathscr{W}_C) = n-i$, $i=1,\dots,n$.] The varieties $\mathrm{envl}^1(\mathscr{W}_C)$, $\mathrm{envl}^{n-1}(\mathscr{W}_C)$ are known as the *dual variety* and the *dual curve* of C, respectively. We denote $\mathrm{envl}^{n-1}(\mathscr{W}_C) = C^\vee$ which is given by the local mapping:

$$\phi^\vee = (W_0(\phi_0, \dots, \phi_n) : \dots : W_n(\phi_0, \dots, \phi_n)),$$

where $W_i(\phi_0, \dots, \phi_n) = (-1)^i W(\phi_0, \dots, \phi_{i-1}, \phi_{i+1}, \dots, \phi_n)$. By an easy calculation we have

$$\sum_{i=0}^{n} \phi_i^{(k)} \cdot W_i^{(l)} \equiv 0, \quad 0 \leqq k+l \leqq n-1,$$

$$\sum_{i=0}^{n} \phi_i^{(k)} \cdot W_i^{(l)} = (-1)^n \cdot W(\phi_0, \dots, \phi_n), \quad k+l=n,$$

from which we have

$$W(\phi_0, \dots, \phi_n) \cdot W(W_0, \dots, W_n) = (-1)^N \cdot W(\phi_0, \dots, \phi_n)^{n+1},$$

so we see that

$$W(\phi_0, \dots, \phi_n)(t) = 0 \leftrightarrow W(W_0, \dots, W_n)(t) = 0.$$

By Cramer's rule, we have

$$\phi_i \cdot W(W_0, \dots, W_n) = (-1)^{N'} \cdot W_i(W_0, \dots, W_n) \cdot W(\phi_0, \dots, \phi_n).$$

This shows the dualities of the correspondence of space curves:

$$\phi(\tilde{C}) = C \leftrightarrow C^\vee = \phi^\vee(\tilde{C}),$$

and bundles

$$(\mathrm{Osc}^i C)^\vee = \mathrm{Osc}^{n-i-1} C^\vee$$

over $\tilde{C}$ and

$$\mathrm{envl}^i(\mathscr{W}_C) = P^\vee(\mathrm{Osc}^{n-i-1} C^\vee),$$

where $P^\vee$ denotes the natural projection into the dual space $\mathbb{P}_n^\vee$.

Since the projection $P^\vee : (\mathrm{Osc}^0 C_0)^\vee = \mathrm{Osc}^{n-1} C_0^\vee = \gamma \to \mathbb{P}_n^\vee$ has everywhere rank $\geqq n-1$, $P^\vee$ has only singularities of type A^k, $k=0,1,\dots$ ($=$ Morin $\Sigma^{1,\overset{k}{\cdots},1}$ singularity). By the singular type, γ is filtered by subbundles as:

$$\mathbb{P}_n^\vee \times \tilde{C} \supset \gamma = \Sigma^0(P^\vee) \supset \Sigma^1(P^\vee) \supset \dots \supset \Sigma^{1,\overset{n-1}{\cdots},1}(P^\vee) \supset \dots,$$

where $\Sigma^{1,\overset{i}{\cdots},1}(P^\vee)$ is the set of points where $P^\vee$ is of type $\Sigma^{1,\overset{j}{\cdots},1}$, $j \geqq i$.

The projection $P^\vee$: $\mathrm{Osc}^{n-i-1} C_C^\vee \to \mathbb{P}_n^\vee$ is given locally by the mapping:

$$(u,t) \in \mathbb{P}_n \times \tilde{C} \to {}^t\begin{vmatrix} W_0 & \dots & W_n \\ W_0^{(n-i-1)} & \dots & W_n^{(n-i-1)} \end{vmatrix}(t)\, u,$$

so we see by an easy induction, that

$$\Sigma^{1,\overset{i}{\cdots},1}(P^\vee)=\mathrm{Osc}^{n-i-1}\,C_0^\vee,$$

$$\Sigma(P^\vee:\Sigma^{1,\overset{i}{\cdots},1}(P^\vee)\to\mathbb{P}_n^\vee)=\Sigma^{1,\overset{i+1}{\cdots},1}(P^\vee),$$

$$P^\vee(\Sigma^{1,\overset{i}{\cdots},1}(P^\vee)-(P^\vee))=P^\vee(\mathrm{Osc}^{n-i-1}\,C_0^\vee)=\Sigma^{1,\overset{i}{\cdots},1}(\mathscr{W}_C),$$

and $P^\vee(\Sigma^{1,\overset{i+1}{\cdots},1}(P^\vee))=\Sigma^{1,\overset{i+1}{\cdots},1}(\mathscr{W}_C)$ is the envelope set of $\Sigma^{1,\overset{i}{\cdots},1}(\mathscr{W}_C)$ foliated by fibres of $\mathrm{Osc}^{n-i-1}\,C_0^\vee$.

Conversely we have also

$$\begin{aligned}P^\vee(\mathrm{Osc}^{n-i-1}\,C^\vee)&=\tan\,\mathrm{envl}^{i+1}\,(\mathscr{W}_C)\\&=\tan\,P^\vee(\bar{\Sigma}^{1,\overset{i+1}{\cdots},1}\,(P^\vee))=P^\vee(\bar{\Sigma}^{1,\overset{i}{\cdots},1}(P^\vee))\\&=\mathrm{envl}^i\,(\mathscr{W}_C),\end{aligned}$$

where tan X denotes the tangent variety of X which is the closure of the union of tangent spaces of X at non-singular points.

Summarizing the facts above, we state the following proposition.

PROPOSITION 2.2.1. *We have sequences*:

$$\mathbb{P}_n^\vee=\mathrm{envl}^0\,(\mathscr{W}_C)\supset\mathrm{envl}^1\,(\mathscr{W}_C)\supset\ldots\supset\mathrm{envl}^{n-1}(\mathscr{W}_C)=C^\vee,$$

$$\mathbb{P}_n=\mathrm{envl}^0(\mathscr{W}_C\,{}^\vee)\supset\mathrm{envl}^1\,(\mathscr{W}_C\,{}^\vee)\supset\ldots\supset\mathrm{envl}^{n-1}(\mathscr{W}_C\,{}^\vee)=C$$

and

$$\begin{aligned}\mathrm{envl}^i(\mathscr{W}_C)&=\tan\,\mathrm{envl}^{i+1}(\mathscr{W}_C)=(\tan)^{n-i-1}\,C^\vee,\\&=P^\vee(\mathrm{Osc}^{n-i-1}\,C^\vee)\end{aligned}$$

and

$$\mathrm{envl}^i(\mathscr{W}_C)^\vee=\mathrm{envl}^{n-i-1}(\mathscr{W}_C\,{}^\vee),$$

for $i=1,\ldots,n-1$.

Next we prove

PROPOSITION 2.2.2.

$$\begin{aligned}\mathrm{envl}(\mathscr{W}_C)&=\mathrm{envl}^1(\mathscr{W}_C)\vee(\mathrm{sing}\;C)^\vee\\&=(\tan)^{n-2}\,C^\vee\cup(\mathrm{sing}\;C)^\vee\\&=\tan(\tan(\ldots(\tan C^\vee))\ldots)\cup(\mathrm{sing}\;C)^\vee\end{aligned}$$

Proof. The inclusion envl $(\mathscr{W}_C)\supset\mathrm{envl}^1\,(\mathscr{W}_C)\cup(\mathrm{sing}\,C)^\vee$ is clear. Suppose $x\in\mathrm{envl}\,(\mathscr{W}_C)-(\mathrm{sing}\;C)^\vee$. Then the dual hyperplane has a contact with C_{reg} of order $\geqq 2$ at $\phi(t)$. Since the matrix $\begin{vmatrix}\phi_0 & ,\ldots, & \phi_n\\ \phi_0^{(1)} & ,\ldots, & \phi_n^{(1)}\end{vmatrix}(t)$ has rank 2, x is in the dual space of the line spanned by $\phi(t)$, $\phi^{(1)}\,(t)$. The closure of the union of those dual spaces is precisely the set $\mathrm{envl}^1\,(\mathscr{W}_C)$. So the converse of the inclusion holds. The other part of the statement follows from Proposition 2.2.1.

The structure of $\Sigma(\mathscr{W}_C)\cap(C-C_0)^\vee$ is more complicated and depending on the degeneracy of $W(\phi_0,\ldots,\phi_n)$. But we will not discuss this any further here.

We remark that all singular subsets above are defined only by the topological properties of the web $\mathscr{W}_C$, $\mathscr{W}_{C^\vee}$, because order of contact of subspace with $C, C^\vee$ is a topological quantity which can be recovered by the topological structure of $\mathscr{W}_C$, $\mathscr{W}_{C^\vee}$.

Finally to analyze the whole singular set $\Sigma(\mathscr{W}_C) = \text{envl}\,(\mathscr{W}_C)\ \text{degn}\,(\mathscr{W}_C)$ in a way similar to the above, we define the secant variety:

$$\sec^n(C) = \overline{\{P(x_1, \ldots, x_n) | x_i \in C \text{ are all distinct and } \dim P(x_1, \ldots, x_n) \leqq n-2\}}.$$

Then by definition we have

$$\text{degn}\,(\mathscr{W}_C) = (\sec^n C)^\vee.$$

Of course we can define a filteration of $\sec^n C$ by the degree of degeneracy in the same manner as $\text{envl}^i\,(\mathscr{W}_C)$. However, the author does not know whether there exists any duality like Propositions 2.2.1 and 2.2.2, between subsets of $\text{degn}\,(\mathscr{W}_C)$ and $\text{degn}\,(\mathscr{W}_{C^\vee})$ nor what the set $(\sec^n C)^\vee$ is.

For a point $p \in \mathbb{P}_n - C$, the *normal bundle of the projection* π_p of C from p is defined by

$$\mathscr{N}_p = \pi_p^* T\mathbb{P}_{n-1}/TC.$$

The structure of $\mathscr{N}_p$ is completely translated into the geometry of the hyperplane section $p^\vee \cdot \mathscr{W}_C$. A purely geometric approach might be helpful to understand $\mathscr{N}_C$. In the papers [17, 18, 20], very interesting problems are discussed on $\mathscr{N}_p$.

2.3. *Graf–Sauer's theorem, quasiproduct*

In the following sections, we consider the case that the curve $C \subset \mathbb{P}_n$ is of degree $\leqq n+1$. This is exceptional in Theorem 2.1.3.

Here we refer to

THEOREM 2.3.1. (*Graf–Sauer* [7]) *Let* (C_i, x_i), $i = 1, 2, 3$, *be germs of non-singular projective curves in* $\mathbb{P}_2$ *with* $x_i \in x^\vee (x \in \mathbb{P}_2^\vee)$ *all distinct and let* $\mathscr{W}$ *be the* 3*-web generated by* (C_i, x_i), $i = 1, 2, 3$. *Then* $\mathscr{W}$ *is hexagonal at* x *if and only if* (C_i, x_i) *are germs of a common cubic curve* C.

The proof the theorem and beautiful pictures of hexagonal 3-webs of cubic curves are found in [7].

This theorem hints that the web structure of $\mathscr{W}_C$ of cubic curves is everywhere homogeneous off the singular set $\Sigma(\mathscr{W}_C)$, and so may admit many topological symmetries other than its projective symmetries.

In another point of view, it is known that cubic curves, possibly singular, admit group structure on their smooth parts. This structure is, as well known, intrinsically implied by Abel's theorem, which implies also the hexagonality of the webs. These relations are summarized and generalized in [11]. Now we recall some results on these subjects, which is preliminary for the forthcoming sections.

A *symmetric quasigroup* is a set E with a binary composition law $E \times E \to E$: $(x, y) \to x \circ y$ with the condition:

$$x \circ y = y \circ x, \quad x \circ (x \circ y) = y.$$

In other words, the law is defined by a subset of relations $L_E \subset E \times E \times E$, invariant under the permutations of three entries such that the projections P_i: $L_E \to E \times E$ forgetting the i-th factors are bijective: $x \circ y = z$ if $(x, y, z) \in L_E$. If E is an analytic manifold and L_E is an analytic hypersurface we say E is an *analytic quasigroup*.

We introduce a new composition law $\cdot$ defined by

$$x \cdot y = u \circ (x \circ y),$$

for a base point (unit) $u \in E$. Then

$$(x, y, z) \in L_E \Leftrightarrow x \cdot (y \cdot z) = (x \cdot y) \cdot z = u \circ u.$$

We call E an *Abelian symmetric quasigroup* if the new composition $\cdot$ makes E an Abelian group. Then, for any $u' \subset E$, the corresponding composition is again Abelian (see [23]).

Let $C \subset \mathbb{P}_2$ be an irreducible cubic curve in the projective plane and C_{reg} be its smooth part. The relation $L_C \subset C^3_{\mathrm{reg}}$ is defined as

$$(x, y, z) \in L_C \Leftrightarrow x, y, z \in C_{\mathrm{reg}} \text{ are collinear.}$$

We will see that L_C is a non-singular surface if C is a non-singular curve, i.e. an elliptic curve. The group structure of the cubic curve C_{reg} is defined as above with this symmetric quasigroup.

If C is reducible, i.e. contains a line L as an irreducible component, then $x \circ y$ is not defined for any $x, y \in L$. However, we can define the group structure on it. (For a space curve $C \subset \mathbb{P}_n$ of degree $n+1$, we can define an n-nary symmetric quasiproduct.)

The web structure of cubic curve is equivalent to the symmetric quasigroup structure of the curve, which is the geometry of the surface $L_C \subset C^3_{\mathrm{reg}}$, foliated by the coordinate lines in C^3_{reg} forming a 3-web of codimension 1 on L_C.

4. *Non-singular curves of degree $n, n+1$*

In this section we study the structure of the web $\mathscr{W}_C$ for the case that C is a non-singular curve of degree $n, n+1$ in projective n space.

It is known that such a curve is a rational or an elliptic curve. In general, any curve in $\mathbb{P}_n$ is obtained by projecting a normal curve in projective space of dimension $\geqq n$. By a generalization of Propositions 2.1.1 and 2.1.2, the web generated by the projection is given by the plane section of the web of the normal curve. So, in this section, we restrict ourselves to the case of rational or elliptic normal curves in projective n space.

A. *The rational normal curve.* This curve is projectively equivalent to the twisted curve which is given by the Veronese imbedding:

$$v(t) = (1 : t : t^2 : \ldots : t^n) : \mathbb{P}_1 \subsetneq \mathbb{P}_n.$$

Let $t_i, i = 1, \ldots, n$ be distinct points. Then the intersection of the dual hyperplanes $W(t) = \bigcap_{i=1}^{n} v(t_i)^\vee$ in $\mathbb{P}^*_n$ is the dual of the image of the matrix

$$\begin{pmatrix} 1, t_1, t_1^2, \ldots, & t_1^n \\ \vdots & \vdots \\ 1, t_n, t_n^2, \ldots, & t_n^n \end{pmatrix}$$

of which the $n \times n$ minors give the Plücker coordinates of the image. We can calculate this as

$$v^\vee = (\sigma_n \Pi : \sigma_{n-1} \Pi : \ldots : \Pi),$$

where $\Pi(t) = \prod_{i<j} (t_i - t_j)$ and σ_i is the basic symmetric polynomial $\sigma_i(t) = \Sigma t_{s_1} \ldots t_{s_i}$, where $\{s_1, \ldots, s_i\}$ runs over all i point subsets of $\{1, \ldots, n\}$. By this form we see that $v^\vee$: $\mathbb{P}^n_1 - \Delta \to \mathbb{P}_n$ extends analytically to the mapping $v^\vee$: $\mathbb{P}^n_1 \to \mathbb{P}_n$: $v^\vee = (\sigma_n : \sigma_{n-1} : \ldots : \sigma_1 : 1)$ (the quotient map) so that

$$v^\vee(\{t_1, \ldots, t_n) \in \mathbb{P}^n_1 | t_i = t'_i\}) = v(t'_i)^\vee \subset \mathbb{P}^\vee_n.$$

So $\mathscr{W}_C$ is octahedral.

A homeomorphism of $\mathscr{W}_C^{\vee}$ lifts to an equivariant homeomorphism of $\mathbb{P}_1^n$ preserving the octahedral n-web with leaves $L_{t_i'} = \{(t_1, \ldots, t_n) \in \mathbb{P}_1^n \mid t_i = t_i'\}$, $t_i' \in \mathbb{P}_1$, $i = 1, \ldots, n$, so of the form $h \times \ldots \times h$ with a homeomorphism h of $\mathbb{P}_1$.

Conversely, any homeomorphism h of $\mathbb{P}_1$ induces a homeomorphism h^n of $\mathbb{P}_1^n$ hence a homeomorphism of the quotient space $(\mathbb{P}_n, \mathscr{W}_C^{\vee})$. By this correspondence, we have:

Proposition 2.4.1. *Let $C \subset \mathbb{P}_n$ be a rational normal curve in projective n-space. Then*

$$\text{Homeo}\,(\mathscr{W}_C^{\vee}) = \text{Homeo}\,(\mathbb{P}_1)$$

and

$$\text{Homeo}\,(\mathscr{W}_C^{\vee}) \cap \text{PGL}(n+1, \mathbb{C}) = \text{Aut}\,(\mathbb{P}_1),$$

where Homeo $(\mathscr{W}_C^{\vee})$ *denotes the group of homeomorphism of* $\mathscr{W}_C^{\vee}$.

Let

$$\Delta^i = \{(t_1, \ldots, t_n) \in \mathbb{P}_1^n \mid i+1 \text{ of } t_j\text{'s are the same}\} \subset \mathbb{P}^n.$$

Then we have

$$v^{\vee}(\Delta^i) = \text{envl}^i(\mathscr{W}_C^{\vee}) \text{ irreducible for } i = 0, \ldots, n-1$$

$$v^{\vee}(\Delta^1) = \text{envl}^1(\mathscr{W}_C^{\vee}) = \Sigma(\mathscr{W}_C^{\vee}),$$

$$v^{\vee}(\Delta^{n-1}) = C^{\vee}: \text{dual curve.}$$

Since $v^{\vee} = (1 : \sigma_1 : \ldots : \sigma_n)$, the restriction $v^{\vee} | \Delta^{n-1}$ is the Veronese imbedding. So the dual curve $C^{\vee}$ is again a rational normal curve.

In the following we refer to a result by Piene [26]:

Proposition 2.4.2. *Let $C \subset \mathbb{P}_n$ be a rational normal curve. Then* envl^i $(\mathscr{W}_C^{\vee})$ *is an irreducible variety of dimension $n-i$ and*

$$\text{degree envl}^i\,(\mathscr{W}_C^{\vee}) = (i+1)\,(n-i),$$

for $i = 0, \ldots, n-1$.

Proof. The degree is presented in [26]. The other statements are seen above.

B. *The elliptic normal curve of degree $n+1$ (non-singular curve of degree $n-1$ with genus 1).* This curve is projectively equivalent to the elliptic normal curve canonically imbedded in $\mathbb{P}_n$. First we recall a classical result on the elliptic curve.

The elliptic curve C is a complex manifold given by the quotient of $\mathbb{C}$ by a non-degenerate lattice $\Lambda = (\omega_1, \omega_2)$. The complex structure of C is determined by the well-known j invariant:

$$j(\lambda) = \frac{4(\lambda^2 - \lambda + 1)^3}{27\lambda^2(\lambda - 1)^2}, \quad \lambda = \frac{\omega_2}{\omega_1}.$$

An imbedding of C into $\mathbb{P}_n$ is given by doubly periodic functions. Here we refer to the paper by Hulek [20] for an explicit construction of and imbedding of C with degree $n+1$:

The Weierstrass σ function is defined by

$$\sigma(z) = z \cdot \prod_{\omega \in \Lambda - 0} \left(1 - \frac{z}{\omega}\right) \exp\left(\frac{z}{\omega} + \frac{z^2}{2\omega^2}\right).$$

With respect to translations by ω_1, ω_2, the following fundamental formula holds:

$$\sigma(z+\omega_i) = -\exp\left[\eta_i\left(z+\frac{\omega_i}{2}\right)\right]\cdot\sigma(z),$$

where η_i is the period constant of the Weierstrass ζ-function.

Case $n \geqq 2$, even. For $p, q \in \mathbb{Z}$, define

$$\sigma_{p,q}(z) = \sigma\left(z - \frac{p\omega_1 + q\omega_2}{n+1}\right),$$

$$\omega = -\exp\left(-\frac{\eta_1}{2}\frac{\eta_2\omega_1}{2}\right), \quad \theta = \exp\left(-\frac{\eta_1\omega_1}{2n+2}\right),$$

and

$$x_m(z) = \omega^m \cdot \theta^{m^2} \cdot e^{m\eta_1 z} \cdot \sigma_{m,0}(z) \cdot \ldots \cdot \sigma_{m,n}(z).$$

Case $n \geqq 3$, odd.

$$\tilde{\sigma}_{p,q}(z) = \sigma\left(z - \frac{p\omega_1 + q\omega_2}{n+1} - \frac{1}{2}\left(\omega_1 + \frac{\omega_2}{n+1}\right)\right),$$

$$\tilde{\omega} = \exp\left[-\tfrac{1}{2}(\eta_1\omega_1 + \eta_2\omega_1)\right]$$

and

$$x_m(z) = \tilde{\omega}^m \cdot \theta^{m^2} \cdot \mathrm{e}^{m\eta_1 z}\tilde{\sigma}_{m,0}(z) \cdot \ldots \cdot \tilde{\sigma}_{m,n}(z).$$

Then we see that $x_{n+1+m}(z) = x_m(z)$ for any integer m, $x_0, \ldots, x_n$ form a basis of $\Gamma(O_C((n+1)O))$, and that the map $v = (x_0 : \ldots : x_n)$: $C \subsetneq \mathbb{P}_n = \mathbb{P}\Gamma(O_C((n+1)O))^*$ is a non-degenerate nornal imbedding of degree $n+1$.

Let $\varepsilon = \exp[(2\pi\sqrt{-1})/(n+1)]$. Then the followings hold:

(1) $$x_i(-z) \sim (-1)^{n+1}x_{-i}(z)$$

(2) $$x_i\left(z - \frac{\omega_1}{n+1}\right) \sim x_{i+1}(z)$$

(3) $$x_i\left(z + \frac{\omega_2}{n+1}\right) \sim e^i \cdot x_i(z),$$

where $\sim$ means that equality holds up to a common nowhere vanishing function independent of i. Then, by (2) and (3), the action of the group $\mathbb{Z}_{n+1} \times \mathbb{Z}_{n+1}$ of translations by $\omega_1/(n+1)$, $\omega_2/(n+1)$ is compatible with the action on $C = \mathbb{C}/\Lambda$ generated by

$$(x_0 : \ldots : x_n) \mapsto (x_n : x_0 : \ldots : x_{n-1})$$
$$(x_0 : \ldots : x_n) \mapsto (x_0 : \varepsilon x_1 : \ldots : \varepsilon^n x_n).$$

By (1) the involution

$$(x_0 : \ldots : x_n) \mapsto (x_0 : x_{-1} : \ldots : x_{-n})$$

induces the involution $z \to -z$ on $C = \mathbb{C}/\Lambda$.

The inverse of the imbedding v is given by the Abelian integral

$$v(z) \mapsto \int_0^z \mathrm{d}z \equiv z \pmod{\Lambda}.$$

Then Abel's theorem says that two divisors

$$\sum_{i=0}^{n} v(a_i), \quad \sum_{i=0}^{n} v(b_i)$$

are linearly equivalent if and only if

$$\sum_{i=0}^{n} a_i \equiv \sum_{i=0}^{n} b_i \pmod{\Lambda}.$$

Let $\mathbb{P}_{n-1}=\{x_0=0\}\subset\mathbb{P}_n$ and $v(C)\cdot\mathbb{P}_{n-1}=\sum_{i=0}^{n} v(p_i)$, $p_i\in C=\mathbb{C}/\Lambda$. This theorem is rephrased as:

$$v(z_i),\ i=0, \ldots, n \textit{ are coplanar}$$

if and only if

$$\sum_{i=0}^{n} z_i \equiv \sum_{i=0}^{n} p_i \pmod{\Lambda}.$$

By the explicit form of $x_m(z)$, we see

$$p_i = \begin{cases} \dfrac{i\omega_2}{n+1}\in C & n\text{: even} \\[2mm] \dfrac{i\omega_2}{n+1}+\dfrac{1}{2}\left(\omega_1+\dfrac{\omega_2}{n+1}\right) & n\text{: odd,} \end{cases}$$

so, in both cases, we have

$$\sum_{i=0}^{n} p_i \equiv 0 \pmod{\Lambda}.$$

Therefore we have proven that:

$v(z_i)$, $z_i\in C=\mathbb{C}/\Lambda$, $i=0, \ldots, n$ *are coplanar if and only if*

$$\sum_{i=0}^{n} z_i \equiv 0 \pmod{\Lambda}.$$

Then we denote the hyperplane $\mathbb{P}_{n-1}$ with $\mathbb{P}_{n-1}\cdot v(C)=v(z_0)+\ldots+v(z_n)$ by $P(z_0, \ldots, z_n)$ and $v^\vee(z_0, \ldots, z_n)=\bigcap_{i=0}^{n} v(z_i)^\vee=P(z_0, \ldots, z_n)^\vee$.

Let

$$L=\{(z_0, \ldots, z_n)\ C^{n+1}|\Sigma z_i\equiv 0 \bmod \Lambda\}\subset C^{n+1}$$

and

$$\Delta^i=\{(z_0, \ldots, z_n)\in C^{n+1}|i+1 \text{ of } z_j\text{s are the same}\}.$$

[$\Delta^n\cap L$ consists of $(n+1)^2$ points.] We see that:

$P(z_0, \ldots, z_n)$ *has contact with* $C_{n+1}=v(C)$ *of order* $\geqq i+1$ *if and only if* $(z_0, \ldots, z_n)\in\Delta^i$,

and the map $v^\vee: L\to\mathbb{P}_n^\vee$ possesses the following properties:

(1) $v^\vee$ is an $n+1$-sheeted covering with branch locus Δ^1 (the quotient map by the symmetry group),

(2) $v^\vee(z_i=a)=v(a)^\vee$,

(3) $\mathscr{W}_{C_{n-1}}$ *is octahedral off the singular set* $\Sigma(\mathscr{W}_{C_{n+1}})$,

(4) $v^\vee(\Delta^i)=\mathrm{envl}^i(\mathscr{W}_{C_{n-1}})$ is irreducible,

(5) $v^\vee(\Delta^{n-1})=$ *the dual curve of* C_{n+1},

(6) $v^{\vee}(\Delta^n)=(n+1)^2$ singular points of $v^{\vee}(\Delta^{n-1})$
$\quad$ = duals of hyperosculating hyperplane of C_{n+1} at $n+1$ torsion points,
(7) $\operatorname{envl}^1(\mathscr{W}_{C_{n+1}})=\Sigma(\mathscr{W}_{C_{n+1}})$ (for the $n+1$ web of L is nowhere degenerate).

Here we refer again to a result from [26].

PROPOSITION 2.4.5. *Let $C_{n+1}\subset\mathbb{P}_n$ be the elliptic normal curve of degree $n+1$. Then $\operatorname{envl}^i(\mathscr{W}_C)$ is an irreducible variety of dimension $n-i$ and*

$$\text{degree } \operatorname{envl}^1\,(\mathscr{W}_C)=(i+1)\,(n+1).$$

Proof. The degree is presented in [26]. The other statements are seen above.

Now we consider the topological symmetry of the web $\mathscr{W}_{C_{n+1}}$ of $\mathbb{P}_n^{\vee}$.

Let $\Lambda=(\omega_1,\omega_2),\Lambda'=(\omega_1',\omega_2')$ be non-degenerate lattices of $\mathbb{C}$ and C_{n+1}, C_{n+1}' be the corresponding elliptic normal curves canonically imbedded in $\mathbb{P}_n$ as above, and suppose that the associated $(n+1)$-webs $\mathscr{W}_{C_{n+1}},\mathscr{W}_{C'_{n+1}}$ are topologically equivalent by a homeomorphism h of $\mathbb{P}_n^{\vee}$. Then h induces a homeomorphism $h^{\vee}\colon C_{n+1}\to C'_{n+1}$ such that $h^{\vee n+1}\colon L_\Lambda\to L_{\Lambda'}$ is a homeomorphism and $(n+1)\cdot h^{\vee}(0)\equiv 0 \bmod \Lambda'$ (Proposition 2.1.2). Composing with the translation $T\colon(\mathbb{C}/\Lambda,h^{\vee}(0))\to(\mathbb{C}/\Lambda',0)$, $T\circ h^{\vee}$ is an isomorphism of $\mathbb{C}/\Lambda$ to $\mathbb{C}/\Lambda'$ as topological groups, hence a real linear isomorphism of the torus group T^2, which we identify naturally with an element of $GL\,(2,\mathbb{Z})$ acting on the lattice $\mathbb{Z}\times\mathbb{Z}\subset\mathbb{R}^2=\mathbb{C}$. So $h^{\vee}$ is a composition of $T\circ h^{\vee}\in GL(2,\mathbb{Z})$ with T^{-1}, $T^{-1}(0)\in\mathbb{Z}_{n+1}\times\mathbb{Z}_{n+1}\subset T^2$. These compositions form a lattice $(\mathbb{Z}_{n+1}\times\mathbb{Z}_{n+1})$-preserving subgroup G of the affine transformation group of T^2. Note that the group G is an extension of $GL\,(2,\mathbb{Z})$ by $\mathbb{Z}_{n+1}\times\mathbb{Z}_{n+1}$:

$$0\to\mathbb{Z}_{n+1}\times\mathbb{Z}_{n+1}\to G\to GL(2,\mathbb{Z})\to 0.$$

From now on we denote G by semi-direct product $GL(2,\mathbb{Z})\ltimes(\mathbb{Z}_{n+1}\times\mathbb{Z}_{n+1})$.

Conversely any linear isomorphism $h^{\vee}\colon\mathbb{C}/\Lambda\to\mathbb{C}/\Lambda'$ preserving the lattice $\mathbb{Z}_{n+1}\times\mathbb{Z}_{n+1}$ induces a homeomorphism h of $\mathbb{P}_n^{\vee}$ such that $h(\mathscr{W}_{C_{n+1}})=\mathscr{W}_{C'_{n+1}}$.

Summarizing the results above, we have

PROPOSITION 2.4.3. *For any elliptic normal curves $C,C'\subset\mathbb{P}_n$ of degree $n+1$, the associated $(n+1)$-webs $\mathscr{W}_C$, $\mathscr{W}_{C'}$, are topologically equivalent.*

Let $C=C'$, i.e. $\Lambda=\Lambda'$. The correspondence above of the group G with homeomorphisms of $\mathscr{W}_C$ to $\mathscr{W}_{C'}$, gives a representation of G in the group Homeo $(\mathbb{P}_n)$. It is easy to see that $G\cap PGL(n+1,\mathbb{C})=\mathbb{Z}_i\ltimes(\mathbb{Z}_{n+1}\times\mathbb{Z}_{n+1})$, where $\mathbb{Z}_i$ is the cyclic subgroup of $GL(2,\mathbb{Z})$ generated by $\begin{pmatrix}0&-1\\1&0\end{pmatrix}$ $(i=4)$ if Λ is square, $\begin{pmatrix}0&-1\\1&1\end{pmatrix}$ $(i=6)$ if Λ is triangular, and $\begin{pmatrix}-1&0\\0&-1\end{pmatrix}$ $(i=2)$ otherwise.

Finally we summarize the results as follows.

PROPOSITION 2.4.4. *Let $C\subset\mathbb{P}_n$ be an elliptic normal curve of degree $n+1$. Then the associated $(n+1)$-web $\mathscr{W}_C$ is hexagonal off the singular set $\Sigma(\mathscr{W}_C)=\operatorname{envl}^1\,(\mathscr{W}_C)$. Furthermore*

$$\text{Homeo }(\mathscr{W}_C)=GL(2,\mathbb{Z})\ltimes(\mathbb{Z}_{n+1}\times\mathbb{Z}_{n+1})$$

and

$$\text{Homeo }(\mathscr{W}_C)\cap PGL(n+1,\mathbb{C})=\mathbb{Z}_i\ltimes(\mathbb{Z}_{n+1}\times\mathbb{Z}_{n+1}),$$

where $i=4$ if C is square, $i=6$ if C is triangular and $i=2$ otherwise, and $\ltimes$ denotes the semi-direct product.

2.5. *Singular cubics*

By projective transformations, a cubic plane curve is isomorphic to one of the normal form in Table 1. We will see the web structure of the curves there by using their group structure. Case (1) is the simplest case of the elliptic normal curves in Section 4, and Cases (2) and (3) are almost the same as (1).

Case (4): Conic and line, $y(x^2+y^2-z^2)=0$.

This curve is a union of the conic $C_1=P_1=\{x^2+y^2-z^2=0\}$ and the line $C_2=L_1=\{y=0\}$. We take their parametrizations:

$$F_1(t)=\left(\frac{1-t^2}{1+t^2}:\frac{-2t}{1+t^2}:1\right),\quad F_2(t)=\left(\frac{t-1}{t+1}:0:1\right),$$

$t\in\mathbb{P}_1$, and we put a group structure $C^*\times\mathbb{Z}_2$ on $C_{\mathrm{reg}}=C_1\cup C_2-(\pm1:0:1)$ as follows.

We can easily see that

(*) $F_1(a)$, $F_1(b)$, $F_2(c)$ *are collinear if and only if* $abc=1$.

Then we define the symmetric quasi-product $\circ$ (see Section 3) by

$$F_1(a)\circ F_1(b)=F_2(c)$$

and the multiplication $\cdot$ on C_{reg} by

$$F_1(a)\cdot F_1(b)=F_1(1)\circ(F_1(a)\circ F_1(b))\in C_1$$
$$F_1(a)\cdot F_2(b)=F_1(1)\circ(F_1(a)\circ F_2(b))\in C_2$$
$$F_2(a)\cdot F_2(b)=(F_2(a)\circ F_1(1))\circ(F_2(b)\circ F_1(1))\in C_1.$$

By computation, we see the formulae

$$F_1(a)\cdot F_1(b)=F_1(ab),$$
$$F_1(a)\cdot F_2(b)=F_2(ab),$$
$$F_2(a)\cdot F_2(b)=F_1(ab),$$

Table 1. Singular cubics

	Group structure	Homeo ($\mathscr{W}_C$)	Homeo ($\mathscr{W}_C$)$\cap PGL(2,\mathbb{C})$
(1) Non-singular cubic (elliptic curve) where $i=4$ if Λ is square, $i=6$ if Λ is triangular, and $i=2$ otherwise	$\mathbb{C}/\Lambda$	$GL(2,\mathbb{Z})\ltimes(\mathbb{Z}_3\times\mathbb{Z}_3)$	$\mathbb{Z}_i\times(\mathbb{Z}_3\times\mathbb{Z}_3)$.
(2) Cuspidal cubic	$\mathbb{C}$	$GL(2,\mathbb{R})$	$\mathbb{C}^*$
(3) Nodal cubic	$\mathbb{C}^*=\mathbb{C}/2\pi i$	$\mathbb{R}^*\times\mathbb{Z}_3$	$\mathbb{Z}_3$
(4) Conic and line	$\mathbb{C}^*\times\mathbb{Z}_2$	$\mathbb{R}^*\ltimes\mathbb{C}^*$	$\mathbb{C}^*$
(5) Conic and tangent	$\mathbb{C}\times\mathbb{Z}_2$	$GL(2,\mathbb{R})\ltimes\mathbb{C}$	$\mathbb{C}^*\ltimes\mathbb{C}$
(6) Triangle	$\mathbb{C}^*\times\mathbb{Z}_3$	$(\mathbb{R}^*\times\sigma_3)\ltimes(\mathbb{C}^*\times\mathbb{C}^*)$	$\sigma_3\ltimes(\mathbb{C}^*\times\mathbb{C}^*)$
(7) Three concurrent line	$\mathbb{C}\times\mathbb{Z}_3$	$(GL(2,R)\times\sigma_3)\ltimes(\mathbb{C}\times\mathbb{C})$	$(\mathbb{C}^*\times\sigma_3)\ltimes(\mathbb{C}\times\mathbb{C})$
(8) Two lines one repeated	—	$\mathbb{Z}_2\ltimes\mathrm{Homeo}\,(\mathbb{P}_1)^2$	$(\mathbb{Z}_2\ltimes)PGL(2,\mathbb{C})^2$
(9) Triple lines	—	—	—

Where σ_3 is the symmetric group of order 3, $\ltimes$ denotes the semi-direct product (see also Proposition 2.4.4) and $\mathbb{C}^*\times\mathbb{C}^*$, $\mathbb{C}\times\mathbb{C}$ are the subgroups of $(\mathbb{C}^*)^3$, $(\mathbb{C})^3$ defined by $abc=1$, $a+b+c=0$, respectively.

which makes C_{reg} the group $C^* \times \mathbb{Z}_2$, and that $p, q, r \in C_{reg}$ *are collinear if and only if* $p \cdot q \cdot r = F_1(1)$.

Let $\mathscr{W}_C$ be the 3-web generated by C, and let h be a homeomorphism of $\mathscr{W}_C$. It is easy to see that h induces a homeomorphism $h^\vee$ of $C_{reg} = \mathbb{C}^* \times \mathbb{Z}_2$ respecting the second terms, by the relation (*).

Let $h^\vee = f \cup g$, where f, g are homeomorphisms of $\mathbb{C}^* = C_1$, $\mathbb{C}^* = C_2$, respectively. By (*) we see that

$$f(a)\, f(b)\, g(c) = 1 \quad \text{if} \quad abc = 1.$$

Define $f' = f/f(1)$ and $g' = g/g(1)$. Then we see that

$$f'(a)\, f'(b)\, g'(c) = 1 \quad \text{if } abc = 1$$

and

$$f'(1) = g'(1) = 1.$$

This relation implies that $f' = g'$ and it is a group homeomorphism of the form:

$$\log f'(z) = \lambda \text{ Re } \log z + i \text{ Im } \log z, \quad \lambda \in \mathbb{R}^*.$$

Since $f = f' \times f(1)$, $g = f' \times g(1)$ and $f(1)^2 g(1) = 1$, $h^\vee$ corresponds to the element $(f', f(1))$ of the semi-direct product $\mathbb{R}^* \rtimes \mathbb{C}^*$. So we have

$$\text{Homeo } (\mathscr{W}_C) = \text{Aut}_{top}\, (\mathbb{C}^*) \rtimes \mathbb{C}^* = \mathbb{R}^* \rtimes \mathbb{C}^*.$$

For the other cases (5)–(7), we can analyze the group structures similarly.

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